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AMBIENT SURGERY AND TANGENTIAL HOMOTOPY QUATERNIONIC PROJECTIVE SPACES

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Introduction. In this paper the word manifold will always mean oriented compact C^∞-manifold. Unless otherwise specified, all homology and cohomology is taken with integral coefficients, and for M^n an n-manifold, $[M] \in H_n(M, \partial M)$ will denote the orientation class of M. A map $f: M \to N$ between n-manifolds is of degree $+1$ if $f_*([M]) = [N]$.

We denote the quaternions by Q, and quaternionic projective n-space, which may be described as the collection of quaternionic lines in Q^{n+1}, by QP_n. The underlying set of QP_n can also be described as

$$\left\{(x_1, \ldots, x_{n+1}) \mid x_i \in Q \text{ for } i = 1, \ldots, n+1; \sum_{i=1}^{n+1} |x_i|^2 = 1\right\}$$

modulo the equivalence relation $(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})$ if and only if for some $s \in Q$, $|s| = 1$, $(x_1, \ldots, x_{n+1}) = s \cdot (y_1, \ldots, y_{n+1})$. By $QP_{n-1} \subset QP_n$ we will always mean the copy of QP_{n-1} in QP_n defined by: the class of $(x_1, \ldots, x_{n+1})$ is in $QP_{n-1} \subset QP_n$ if and only if $x_{n+1} = 0$. QP_n is given the structure of a manifold as in [14, §20.3].

$\mathscr{S}^4$ will denote the unit 4-disc bundle over QP_n defined by the standard vector 4-bundle over QP_n, whose total space is $\{(l, v) \mid l \text{ a quaternionic line in } Q^{n+1}, v \text{ an element of } l\}$, and whose projection takes (l, v) to l. $\mathscr{S}_0^3$ will denote the 3-sphere bundle over QP_n associated with $\mathscr{S}^4$. The group of $\mathscr{S}_0^3$ may be reduced to $S^3 =$ the multiplicative group of unit length quaternions, and, since the total space of $\mathscr{S}_0^3$ over QP_n is S^{4n+3}, we see $\mathscr{S}_0^3$ is the universal bundle with fiber S^3 and group the multiplicative group of unit length quaternions for dimensions less than or equal to $4n+2$, see [14, §19].

Given a manifold M with submanifold N, we denote the normal disc bundle of N in M by $\nu(N \subset M)$, and we identify a tubular neighborhood of N in M with the total space of $\nu(N \subset M)$. $\tau(M)$ will denote the tangent bundle of M, and 0_M^n the trivial vector n-bundle over M. A map between manifolds $f: R \to S$ is called tangential if for some integers k, l, $f^*(\tau(S)) \oplus 0_R^k \approx \tau(R) \oplus 0_R^l$.

Statement of results. The first section of this paper is devoted to a discussion of surgery [6], [8], [12]. In particular, we show that in special cases one can do surgery

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to a submanifold N of a manifold M so that the modified manifold is also a submanifold of M. Using this technique we prove

THEOREM 1. *Let P^{m-l} be a simply connected submanifold of a simply connected manifold Q^m, where $l \geqq 3$, $m-l \geqq 5$, and $m-l$ congruent to* 0, 1, *or* 3 *modulo* 4. *Suppose P, Q, and M^m are manifolds without boundary and that $f: M^m \to Q^m$ is a degree* +1 *tangential homotopy equivalence. Then f is homotopic to a map $f': M \to Q$ which is differentiable and transverse regular over $P \subset Q$ and such that*

$$f'|f'^{-1}(P): f'^{-1}(P) \to P$$

is a homotopy equivalence.

We also prove a uniqueness theorem for the $m-l \equiv 0$ modulo 4 case:

THEOREM 2. *Let P and Q be as in Theorem* 1, *with* $m-l \equiv 0 \bmod 4$. *Assume M^{m+1} is an h-cobordism between M_1^m and M_2^m. Let $f: M^{m+1} \to Q$ be a tangential homotopy equivalence which is differentiable and transverse regular over $P \subset Q$. Let $N_i = f^{-1}(P) \cap M_i$ for $i=1, 2$. Assume further that $f|N_i: N_i \to P$ is a homotopy equivalence for $i=1, 2$. Then f is homotopic to a map $f': M^{m+1} \to Q$ such that $f'|M_1^m \cup M_2^m = f|M_1^m \cup M_2^m$, and such that f' is differentiable and transverse regular over $P \subset Q$ and $f'|f'^{-1}(P): f'^{-1}(P) \to P$ is a homotopy equivalence so that $f'^{-1}(P)$ is an h-cobordism between N_1 and N_2.*

In §2 we apply the results of the first section to the study of manifolds of the same tangential homotopy type as QP_n.

DEFINITION. For $n \geqq 2$, $\theta(QP_n)$ is the set of equivalence classes of pairs (M, f) where M is a $4n$-manifold without boundary and $f: M \to QP_n$ is a tangential homotopy equivalence of degree +1 under the relation: $(M, f) \sim (N, g)$ if and only if M and N are h-cobordant under a cobordism C such that there is a tangential homotopy equivalence $h: C \to QP_n$ which restricts to f and g on the proper boundary components.

DEFINITION. θ_n is the group of homotopy n-spheres under the equivalence relation of h-cobordism, as discussed in [6]. We will see that θ_{4n} acts as a group on $\theta(QP_n)$ by connected sum for $n \geqq 2$.

Proofs that the following two definitions are independent of the choices made are given in §2. They depend on the theorems of §1.

DEFINITION. Given $\alpha \in \theta(QP_n)$, we will define $r(\alpha) \in \theta(QP_{n-1})$ by choosing a representative element (M, f) for α such that $f: M \to QP_n$ is differentiable and transverse regular over $QP_{n-1} \subset QP_n$ and such that $f|f^{-1}(QP_{n-1}): f^{-1}(QP_{n-1}) \to QP_{n-1}$ is a tangential homotopy equivalence, and setting

$$r(\alpha) = [(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))].$$

DEFINITION. Given $\gamma \in \theta(QP_{n-1})$ we will define $h(\gamma) \in \theta_{4n-1}$ as follows: choose a representative (N, g) for γ such that $g: N \to QP_{n-1}$ is differentiable. Then $g^*(\mathscr{S}_0^3)$ is a homotopy sphere. We define $h(\gamma) = [g^*(\mathscr{S}_0^3)] \in \theta_{4n-1}$.

If for $\lambda \in \theta_{4n}$ we let $c(\lambda)=\lambda\cdot[(QP_n, \text{identity})]$, where the dot indicates the group action of θ_{4n} in $\theta(QP_n)$, we obtain a sequence

$$\theta_{4n} \xrightarrow{c} \theta(QP_n) \xrightarrow{r} \theta(QP_{n-1}) \xrightarrow{h} \theta_{4n-1}.$$

In §2 we prove

THEOREM 3. (1) *image of* r = *kernel of* h,
(2) *if* $r(\alpha)=r(\beta)$, *then* $\alpha=\lambda\cdot\beta$ *for some* $\lambda \in \theta_{4n}$.

This theorem gives us an inductive geometric procedure whereby representatives for all elements of $\theta(QP_n)$ may be constructed from elements of $\theta(QP_{n-1})$. We conclude §2 with this construction.

In §3 we investigate $\theta(QP_2)$ as a starting point for inductions. We prove

THEOREM 4. $\theta(QP_2)$ *contains at most two elements, with representatives given by* (QP_2, *identity*) *and* ($QP_2 \# \Sigma^8$, id'), *where* Σ^8 *is the nonstandard homotopy* 8-*sphere, and* id' *the obvious homeomorphism* $QP_2 \# \Sigma^8 \to QP_2$.

Note. This and Theorem 3 together imply that the number of elements in $\theta(QP_n)$ is finite for all $n \geqq 2$, since θ_n is finite for all $n \geqq 8$ [6].

DEFINITION. A map between manifolds, $h\colon M \to N$, is *induced by a combinatorial equivalence* if for some C^∞ triangulations $\tau_1\colon K_1 \to M$, $\tau_2\colon K_2 \to N$, and some combinatorial equivalence of simplicial complexes $c\colon K_1 \to K_2$, $h=\tau_2 \circ c \circ \tau_1^{-1}$.

The final result of this paper is

THEOREM 5. *For* $n \geqq 2$, *any representative* (M, f) *of an element of* $\theta(QP_n)$ *has the property that* $f\colon M \to QP_n$ *is homotopic to a map* $h\colon M \to QP_n$ *induced by a combinatorial equivalence. Thus, for* $n \geqq 2$, *all* 4n-*manifolds of the tangential homotopy type of* QP_n *are combinatorially equivalent.*

Note. The restriction to manifolds of the tangential homotopy type of QP_n is essential. Hsiang [4] has shown that for any $n \geqq 2$ there exist infinitely many distinct manifolds of the homotopy type of QP_n all with different rational Pontrjagin classes. These manifolds are all combinatorially distinct.

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1. **Ambient surgery.** Corresponding to the assumptions of Theorems 1 and 2, there are two cases in which we will show that surgery can be done within an ambient manifold:

Case (a). P^{m-l} a submanifold of Q^m of codimension l where $l \geqq 3$, $m-l \geqq 5$, and $m-l \not\equiv 2 \bmod 4$. Suppose P and Q are both simply connected, and that P, Q, and M^m are manifolds without boundary. Let $f\colon M \to Q$ be a degree $+1$ tangential homotopy equivalence. By the work of Thom [16] we can deform f by a homotopy so that it becomes differentiable and transverse regular over $P \subset Q$. In this case we will do surgery to $f^{-1}(P)$ to prove that f can be deformed by a series of homotopies to a map $f'\colon M \to Q$ such that f' is transverse regular over $P \subset Q$ and

$$f'|f'^{-1}(P)\colon f'^{-1}(P) \to P$$

is a homotopy equivalence.

Case (b). Assume P^{m-l} and Q^m are as in Case (a), but with $m-l \equiv 0 \bmod 4$. Suppose M^{m+1} is an h-cobordism between M_1^m and M_2^m and $f\colon M^{m+1} \to Q$ is a tangential homotopy equivalence. Then $f|M_i^m\colon M_i^m \to Q$ is a homotopy equivalence for $i=1, 2$. Again we may up to homotopy assume f differentiable and transverse regular over $P \subset Q$. We assume f is of this form, and further that if $N_i = f^{-1}(P) \cap M_i$ then $f|N_i\colon N_i \to P$ is a homotopy equivalence for $i=1, 2$. In this case we will do surgery to $f^{-1}(P)$ to prove that $f\colon M^{m+1} \to Q$ can be deformed by a series of homotopies, all of which are stationary on ∂M^{m+1}, to a map $f'\colon M^{m+1} \to Q$ which is differentiable and transverse regular over $P \subset Q$ and such that $f'^{-1}(P)$ is an h-cobordism between N_1 and N_2.

Recall the following lemma from [16, p. 67].

LEMMA 1.1. *Given $f\colon M^m \to Q$ differentiable and transverse regular over $P \subset Q$ as in Case* (a), *if we put $N = f^{-1}(P)$ then $f|N\colon N \to P$ is a map of degree* ± 1, *and thus N may be oriented so that $f|N$ is a map of degree* $+1$.

In dealing with both Cases (a) and (b) from now on, we will assume $f\colon M \to Q$ is differentiable and transverse regular over $P \subset Q$, and denote $f^{-1}(P)$ by N and $f|N\colon N \to P$ by $g\colon N \to P$.

We now make the inductive assumption that $g\colon N \to P$ is such that $g_*\colon H_r(N) \to H_r(P)$ is an isomorphism for all $r<k$, where $0 \leqq k \leqq (m-l)/2$. We will want to do surgery to N to modify $H_k(N)$. By the transversality of f to $P \subset Q$, we have $\nu(N \subset M) \approx g^*(\nu(P \subset Q))$ and thus the assumption that f is tangential implies $g\colon N \to P$ is tangential. In Case (a) Lemma 1.1 implies g is a map of degree $+1$, so by [12] we see $g_*\colon H_k(N) \to H_k(P)$ is onto, and all classes in the kernel of this map are spherical.

For Case (b), the map $g_*\colon H_i(N) \to H_i(P)$ is always onto, and since $g_*\colon \pi_i(N) \to \pi_i(P)$ is an isomorphism for all $i<k$, we have that all classes in the kernel of $g_*\colon H_k(N) \to H_k(P)$ are spherical see [12].

In either case, let $\alpha \in H_k(N)$ be an element of kernel $(g_*\colon H_k(N) \to H_k(P))$. Since α is spherical and $k \leqq (m-l)/2$, $m-l \geqq 5$, we may represent α by an imbedding $i\colon S^k \to N$. In Case (b) it is clear we may choose i so that $i(S^k) \cap \partial N = \varnothing$. We will assume that α is so represented, and further that $\nu(i(S^k) \subset N)$ is trivial. It should be noted that $i\colon S^k \to N$ is nullhomotopic in M, and thus

$$\tau(M)|i(S^k) \approx 0^m \approx \nu(i(S^k) \subset N) \oplus \nu(N \subset M)|i(S^k).$$

Hence $\nu(i(S^k) \subset N)$ is stably trivial, and this further condition is no restriction for $k<(m-l)/2$. In the middle dimension $k=(m-l)/2$ we must show that kernel

($g_*: H_k(N) \to H_k(P)$) can be reduced to zero by doing surgery only on classes $\alpha \in H_k(N)$ which satisfy this additional condition.

The idea of the following work is to attach a $(k+1)$-disc to N in M via the map i, and, by thickening this disc, to get a spherical modification of N to a new submanifold N' of M and to deform the map $f: M \to Q$ by a homotopy to a map f_1 such that $f_1^{-1}(P) = N'$. The following lemma is readily seen, and its proof is omitted.

LEMMA 1.2. *The map $f: M \to Q$ is homotopic to a map $f_1: M \to Q$ such that $f_1^{-1}(P) = N = f^{-1}(P)$, f_1 is differentiable and transverse regular over $P \subset Q$, and f_1 maps a neighborhood of $i(S^k)$ in N to a point $p \in P$.*

Henceforth we will assume $f: M \to Q$ has the properties described for f_1 in Lemma 1.2.

By the transversality of f we know $\nu(N \subset M) \approx g^*(\nu(P \subset Q))$, so, by choosing a framing for $\nu(P \subset Q)|p$, where $p = f \circ i(S^k)$, we get a framing $\nu_1, \ldots, \nu_l$ for $\nu(N \subset M)|i(S^k)$. We identify $\nu(N \subset M)$ with a tubular neighborhood of N in M and push $i(S^k)$ out into $M - N$ by a map $i_1: S^k \times I \to M$ defined by $i_1(x, \tau) = (i(x), \tau \cdot \nu_1)$. Since $f \circ i: S^k \to P$ is nullhomotopic and $f: M \to Q$ is a homotopy equivalence, the class of $i_1|(S^k \times (1))$ is null in $\pi_k(M)$. The following crucial proposition asserts this holds also in $M - N$.

PROPOSITION 1.1. *$i_1|(S^k \times (1))$ is a nullhomotopic imbedding of S^k in $M - N$.*

Proof. The proof of this proposition will be in several parts.

(1) If $k = 0$, we have S^k is just $S^0 = 2$ points. Since $N \cap i_1(S^0 \times (1)) = \varnothing$, and M is path connected, we may choose a differentiable arc $h: I \to M$ connecting the points of $i_1(S^0 \times (1))$, transverse regular over N. In Case (b), $i_1(S^0 \times (1)) \cap \partial M$ can be assumed empty, and we may choose $h: I \to M$ so that $h(I) \cap \partial M = \varnothing$. The condition that h is transverse to N implies $h(I) \cap N = \varnothing$ since codimension of N in M is $l > 1$, so $i_1(S^0 \times (1))$ is nullhomotopic in $M - N$.

(2) For $k = 1$ a similar proof applies. That is, we take a map $h: D^2 \to M$ such that $h|\partial D^2 = i_1|(S^1 \times (1))$ which is transverse to N. As above h can be chosen so that $h(D^2) \cap \partial M = \varnothing$ in case (b). Transversality again implies $h(D^2) \cap N = \varnothing$ since codimension of N in M is $l > 2$.

(3) For $k \geqq 2$, the following lemma is applied to show the proposition. In applying the lemma, we note that $f \circ i_1|(S^k \times (1))$ is clearly nullhomotopic in $Q - P$.

LEMMA 1.3. *Suppose $k \geqq 2$ and $g_*: H_p(N) \to H_p(P)$ is an isomorphism for all $p < k$. Then if $f' = f|M - N$, $f'_*: \pi_p(M - N) \to \pi_p(Q - P)$ is an isomorphism for $p \leqq k + l - 2$.*

Proof. We first prove this for Case (a). f is transverse regular over $P \subset Q$, so by [16] we may choose a tubular neighborhood $T(P)$ of P in Q so that $f^{-1}(T(P)) \equiv T(N)$

is a tubular neighborhood of N in M. We define $M' = M -$ interior of $T(N)$; $Q' = Q -$ interior of $T(P)$. Then M' and Q' are manifolds with boundary, and $H_p(M-N) \approx H_p(M')$, $H_p(Q-P) \approx H_p(Q')$. Consider the diagram:

$$
\begin{array}{ccc}
H_p(M-N) & \xrightarrow{f'_*} & H_p(Q-P) \\
p_1 \uparrow & & \uparrow p_2 \\
H_p(M') & \xrightarrow{h_*} & H_p(Q') \\
\cap\,[M'] \uparrow & & \uparrow \cap\,[Q'] \\
H^{m-p}(M', \partial M') & \xleftarrow{h_*} & H^{m-p}(Q', \partial Q') \\
E_1 \uparrow & & \uparrow E_2 \\
H^{m-p}(M, T(N)) & \xleftarrow{\tilde{f}_*} & H^{m-p}(Q, T(P)) \\
i_1 \uparrow & & \uparrow i_2 \\
H^{m-p}(M, N) & \xleftarrow{f^*} & H^{m-p}(Q, P)
\end{array}
$$

where the maps are the obvious inclusions or restrictions of f. All of the vertical maps are isomorphisms. By naturality of the cap product and the fact that h_* has degree $+1$, $h_* \circ (\cap\,[M']) \circ h^* = \cap\,[Q']$. Hence f^* is a monomorphism, and f'_* is an epimorphism. Further, if f^* is an isomorphism, then so is f'_*.

Consider the commutative diagram:

$$
\begin{array}{ccccccccc}
H^{m-p-1}(Q) & \longrightarrow & H^{m-p-1}(P) & \longrightarrow & H^{m-p}(Q, P) & \longrightarrow & H^{m-p}(Q) & \longrightarrow & H^{m-p}(P) \\
f^* \downarrow & & g^* \downarrow & & f^* \downarrow & & f^* \downarrow & & g^* \downarrow \\
H^{m-p-1}(M) & \longrightarrow & H^{m-p-1}(N) & \longrightarrow & H^{m-p}(M, N) & \longrightarrow & H^{m-p}(M) & \longrightarrow & H^{m-p}(N).
\end{array}
$$

From this and the 5-lemma we conclude that $f^*: H^{m-p}(Q, P) \to H^{m-p}(M, N)$ is an isomorphism whenever both $g^*: H^{m-p-1}(P) \to H^{m-p-1}(N)$ and $g^*: H^{m-p}(P) \to H^{m-p}(N)$ are.

The lemma of Novikov [7, p. 10] implies these conditions hold if

$$g_*: H_{p+1-l}(N) \to H_{p+1-l}(P) \quad \text{and} \quad g_*: H_{p-l}(N) \to H_{p-l}(P)$$

are isomorphisms. Thus $f^*: H^{m-p}(Q, P) \to H^{m-p}(M, N)$, and hence

$$f'_*: H_p(M-N) \to H_p(Q-P),$$

is an isomorphism for $p+1-l<k$, or $p \leqq k+l-2$.

Since $k \geqq 2$, $\pi_1(N) \approx \pi_1(P) = 0$, and, by assumption, $\pi_1(M) \approx \pi_1(Q) = 0$, so by Van Kampen's Theorem, $\pi_1(M-N) \approx \pi_1(Q-P) = 0$. We have shown

$$f'_*: H_p(M-N) \to H_p(Q-P)$$

is onto for all p, so by Whitehead's Theorem we now see $f'_*\colon \pi_p(M-N) \to \pi_p(Q-P)$ is an isomorphism for all $p \leqq k+l-2$. This concludes the proof of Lemma 1.3 and of Proposition 1.1 for Case (a).

Recall that in Case (b) M is assumed to be an h-cobordism between M_1 and M_2, $N_1 = N \cap M_1$ and $f|M_1\colon M_1 \to Q$ and $f|M_2\colon M_2 \to Q$ are both assumed to be homotopy equivalences.

By the lemma for Case (a), $f'_*\colon H_p(M_1-N_1) \to H_p(Q-P)$ is an isomorphism for all p. Hence $f'_*\colon H_p(M-N) \to H_p(Q-P)$ is onto for all p, and inclusion$_*$: $H_p(M_1-N_1) \to H_p(M-N)$ is one-to-one for all p. Thus $f'_*\colon H_p(M-N) \to H_p(Q-P)$ is an isomorphism if inclusion$_*$: $H_p(M_1-N_1) \to H_p(M-N)$ is onto.

Choose disc bundle structures on tubular neighborhoods $T(N_1)$ and $T(N)$ of N_1 in M_1 and N in M respectively so that the inclusion gives a bundle map $T(N_1) \to T(N)$. Then the naturality of the Thom isomorphism implies that if

$$\text{incl}_*\colon H_p(N_1) \to H_p(N)$$

is an isomorphism, then so is $\text{incl}_*\colon H_{p+l}(T(N_1), \partial T(N_1)) \to H_{p+l}(T(N), \partial T(N))$.

By excision, $H_{p+l}(T(N_1), \partial T(N_1)) \approx H_{p+l}(M_1, M_1-N_1)$ and $H_{p+l}(T(N), \partial T(N)) \approx H_{p+l}(M, M-N)$. Also, if $g_*\colon H_p(N) \to H_p(P)$ is an isomorphism so is incl_*: $H_p(N_1) \to H_p(N)$, so for $p<k$ we have isomorphisms as indicated in the following diagram:

$$\begin{array}{ccccccccc}
H_{p+l}(M_1) & \to & H_{p+l}(M_1, M_1-N_1) & \to & H_{p+l-1}(M_1-N_1) & \to & H_{p+l-1}(M_1) & \to & H_{p+l-1}(M_1, M_1-N_1) \\
\downarrow \approx & & \downarrow \approx & & \downarrow & & \downarrow \approx & & \downarrow \approx \\
H_{p+l}(M) & \to & H_{p+l}(M, M-N) & \to & H_{p+l-1}(M-N) & \to & H_{p+l-1}(M) & \to & H_{p+l-1}(M, M-N)
\end{array}$$

By the 5-lemma we conclude $\text{incl}_*\colon H_{p+l-1}(M_1-N_1) \to H_{p+l-1}(M-N)$ is an isomorphism for $p<k$, and then $\text{incl}_*\colon H_i(M_1-N_1) \to H_i(M-N)$ is an isomorphism for $i \leqq k+l-2$. Thus we see $f'_*\colon H_p(M-N) \to H_p(Q-P)$ is an isomorphism for $p \leqq k+l-2$, and, as in the proof of Case (a), $M-N$ and $Q-P$ are simply connected, so we again apply Whitehead's Theorem to show $f'_*\colon \pi_p(M-N) \to \pi_p(Q-P)$ is an isomorphism for $p \leqq k+l-2$. This completes the proof of Lemma 1.3 and of Proposition 1.1.

We now identify $\{x \in D^{k+1} \mid |x| \geqq 1/2\}$ with $S^k \times I$, and define a map

$$h\colon \{x \in D^{k+1} \mid |x| \geqq 1/2\} \to M$$

to be this identification composed with the above defined map i_1. Since f is transverse to $P \subset Q$ and carries $i_1(S^k \times (0))$ to $p \in P$, $f \circ h$ maps $S^k_{1/2} = \{x \in D^{k+1} \mid |x| = 1/2\}$ to a single point $x \in Q-P$. Denote $\{x \in D^{k+1} \mid |x| \leqq 1/2\}$ by $D^{k+1}_{1/2}$ so that

$$\partial D^{k+1}_{1/2} = S^k_{1/2}.$$

LEMMA 1.4. *There is a map* $h'\colon D^{k+1}_{1/2} \to M-N$ *such that* $h'|S^k_{1/2} = h|S^k_{1/2}$, *and* $f \circ h'\colon (D^{k+1}_{1/2}, S^k_{1/2}) \to (Q-P, x)$ *is nullhomotopic. Thus* $f\colon M \to Q$ *can be deformed by a homotopy so that we may assume* $f \circ h'(D^{k+1}_{1/2}) = x$.

Proof. Consider the diagram:

$$\begin{array}{ccccc} & \pi_{k+1}(M-N) & \xrightarrow{j_1} & \pi_{k+1}(M-N,\, h(S^k_{1/2})) & \xrightarrow{\partial} \pi_k(h(S^k_{1/2})) \\ & \Big\downarrow f_* & & \Big\downarrow f_* & \\ 0 \longrightarrow & \pi_{k+1}(Q-P) & \xrightarrow{j_2} & \pi_{k+1}(Q-P,\, x) & \longrightarrow 0. \end{array}$$

$h|S^k_{1/2}\colon S^k_{1/2} \to M-N$ is nullhomotopic, so there is a map $h_1\colon D^{k+1}_{1/2} \to M-N$ which extends h. Suppose $h_1\colon (D^{k+1}_{1/2}, S^k_{1/2}) \to (M-N, h(S^k_{1/2}))$ represents an element $\alpha \in \pi_{k+1}(M-N, h(S^k_{1/2}))$. Let $\gamma = j_1 \circ (j_2 f_*)^{-1} \circ f_*(\alpha)$. Then a representative of the class $\alpha + (-\gamma)$ defines a map $h'\colon D^{k+1}_{1/2} \to M-N$ which agrees with h on $S^k_{1/2}$ such that $f \circ h'\colon (D^{k+1}_{1/2}, S^k_{1/2}) \to (Q-P, x)$ is nullhomotopic.

We may choose h' within its class to be an imbedding such that the map defined on D^{k+1} by h' on $D^{k+1}_{1/2}$ and h on $\{x \in D^{k+1} \mid |x| \geqq 1/2\}$ is an imbedding [1, Theorem 4.1]. Also, we may now deform f by a homotopy in a tubular neighborhood of $h'(D^{k+1}_{1/2})$ so that $f \circ h'(D^{k+1}_{1/}) = x \in Q-P$.

We will henceforth denote the map $D^{k+1} \to M$ constructed in Lemma 1.4 by the letter h, and will assume that $f\colon M \to Q$ has been put into the form described in that lemma, and in Lemma 1.2.

We have assumed that $\nu(i(S^k) \subset N)$ is trivial, so $\nu(h(D^{k+1} \subset M))|h(S^k)$ is the sum of two bundles, η_1 and η_2, where η_1 is the $(l-1)$-dimensional trivial bundle given by $\nu_2, \ldots, \nu_l$, that is the bundle which results when we split

$$\nu_1 = \nu(h(S^k) \subset h(D^{k+1}))$$

off $\nu(N \subset M)|h(S^k)$, and η_2 is an $(m-l-k)$-dimensional trivial bundle in Case (a) and an $(m+1-l-k)$-dimensional trivial bundle in Case (b). η_1 is, of course, already framed by our choice of $\nu_2, \ldots, \nu_l$, pulled back by f from $\nu(P \subset Q)|p$.

LEMMA 1.5. *With notation as above, there is a framing of $\nu(h(D^{k+1}) \subset M)$ such that restricted to $h(S^k)$ it gives the required framing $\nu_2, \ldots, \nu_l$ of η_1, and some framing of η_2.*

Proof. Framings of $\nu(h(D^{k+1}) \subset M)|h(S^k)$ are represented by elements of $\pi_k(SO(m-k-1))$ in Case (a), and elements of $\pi_k(SO(m-k))$ in Case (b), and extend over $h(D^{k+1})$ if and only if the representative homotopy class is zero. Similarly, elements of $\pi_k(SO(m-l-k))$ in Case (a) and $\pi_k(SO(m+1-l-k))$ in Case (b) represent framings of $\eta_2 = \nu(h(S^k) \subset N)$. There is given a framing of $\eta_1 = \nu(N \subset M)|h(S^k)$. Thus the problem of framing η_2 so that we get a framing of $\nu(h(D^{k+1}) \subset M)|h(S^k)$ which extends over $h(D^{k+1})$ depends only on the map $\pi_k(SO(m-l-k)) \to \pi_k(SO(m-k-1))$ in Case (a) and on $\pi_k(SO(m+1-l-k)) \to \pi_k(SO(m-k))$ in Case (b) induced by inclusion. If the map is onto then η_2 may be framed as needed.

We see this by showing $\pi_k(SO(p)) \to \pi_k(SO(p+1))$ is onto for all relevant values of p. By Steenrod [14, p. 117] this map is onto whenever $k<p$. Since we have by assumption $k \leqq (m-l)/2$, or $k<m-l-k+1$, the proof for Case (b) is complete. Also, for Case (a), if $m-l \equiv 1$ or $3 \bmod 4$, then $k<(m-l)/2$, and $k<m-l-k$, so the proof is also complete in these cases.

Case (a), $m-l \equiv 0 \bmod 4$ remains, since we may have here $k=m-k-l$ or $k=(m-l)/2$. Then k is an even number, and in the fibration sequence

$$\to \pi_{k+1}(S^k) \to \pi_k(SO(k)) \to \pi_k(SO(k+1)) \to \pi_k(S^k) \to \pi_{k-1}(SO(k)) \to$$

it is known that $\pi_k(S^k) \to \pi_{k-1}(SO(k))$ is a monomorphism, so $\pi_k(SO(k) \to \pi_k(SO(k+1))$ is onto, and we are done.

Note. In the above lemma a framing of η_2 which extends over $h(D^{k+1})$ is found. In Case (a) for $m-l \equiv 3 \bmod 4$, we will later need the fact that there are many such framings from which we may choose. Lemma 1.5 shows the framing $\nu_2, \dots, \nu_l$ of η_1 is represented by the zero element of $\pi_k(V_{m-k-1,l-1})$. Different methods of extending this frame over $h(D^{k+1})$ in $\nu(h(D^{k+1}) \subset M)$ are in 1-1 correspondence with elements of $\pi_{k+1}(V_{m-k-1,l-1})$. In the fibration sequence

$$\pi_{k+1}(SO(m-k-1)) \longrightarrow \pi_{k+1}(V_{m-k-1,l-1}) \xrightarrow{\partial} \pi_k(SO(m-l-k))$$
$$\longrightarrow \pi_k(SO(m-k-1))$$

we see $\pi_k(SO(m-k-1))$ is the stable group since $k<m-k-l \leqq m-k-3$, so elements in the image of the map ∂ are just the stably trivial elements of

$$\pi_k(SO(m-l-k)).$$

Consequently, given any framing of η_2 which arises as in Lemma 1.5 for Case (a), $m-l \equiv 3 \bmod 4$, we may modify this frame by any stably trivial element of $\pi_k(SO(m-l-k))$ and still have a frame which arises as in that lemma. The author is grateful to J. Levine for pointing out this possibility. See [5].

We have now attached a disc $h(D^{k+1})$ to $i(S^k)$ in M, and have framed

$$\nu(h(D^{k+1}) \subset M)$$

in such a way as to induce our given frame $\nu_2, \dots, \nu_l$ on $\nu(N \subset M)|h(S^k)$ and some frame on $\nu(i(S^k) \subset N) = \eta_2$. If we look at those frames on $h(D^{k+1})$ which restrict to η_2 on the boundary, and take a tubular neighborhood of these, we get an imbedding

$$\phi\colon D^{k+1} \times D^{m-l-k} \to M^m \text{ in Case (a), or}$$
$$\phi\colon D^{k+1} \times D^{m+1-l-k} \to M^{m+1} \text{ in Case (b)}$$

where $\phi|(S^k \times (0))$ represents the class $\alpha \in H_k(N)$ we wish to kill by surgery. The modified manifold N' we seek is given after suitable rounding of corners by

$$N - \operatorname{int} \phi(S^k \times D^{m-l-k}) \cup \phi(D^{k+1} \times S^{m-l-k-1}) \text{ in Case (a)}$$

and by

$$N - \operatorname{int} \phi(S^k \times D^{m+1-l-k}) \cup \phi(D^{k+1} \times S^{m-l-k}) \text{ in Case (b).}$$

We now want to find a homotopy of $f: M \to Q$ to a map $f_1: M \to Q$ which is differentiable and transverse regular over $P \subset Q$ and such that $f_1^{-1}(P)$ is "isotopic" to the modified manifold N' described above. Since bundle maps can be carried along over homotopies, we will have that f_1 is a tangential map, and so will have recovered the original conditions and be able to proceed inductively. Our method is similar to that of Haefliger [2].

Recall that $h: D^{k+1} \to M$ starts along the vector field ν_1 in $\nu(N \subset M)|i(S^k)$. We extend this imbedding of D^{k+1} in M to an imbedding of D_2^{k+1} = the $(k+1)$-disc of radius 2 in M by mapping along the vector field $-\nu_1$ in a tubular neighborhood of N in M. Denote the resulting map by $h: D_2^{k+1} \to M$. The framing we have chosen for $\nu(h(D^{k+1}) \subset M)$ extends over $\nu(h(D_2^{k+1}) \subset M)$.

In the lemmas above, we have modified the map f so that f maps a tubular neighborhood of $i(S^k)$ in N to $p \in P$, is transverse to P, and $f \circ h | D_{1/2}^{k+1}$ has image a point $x \in Q - P$. It is now clear one can deform f by a homotopy and choose coordinates for a tubular neighborhood of $h(D_2^{k+1})$ in M of the form $(x, y, z) \in D_2^{k+1} \times D_2^r \times D^{l-1}$, where $r = m - l - k$ in Case (a) and $r = m + 1 - l - k$ in Case (b), such that on this neighborhood the map f is defined by the following composition:

Let $s_1: R \times D^{l-1} \to [-1, 1] \times D^{l-1}$ be identity on $[-1, 1] \times D^{l-1}$, $s_1(x, p) = (-1, p)$ for $x \leqq -1$, and $s_1(x, p) = (1, p)$ for $x \geqq 1$.

$s_2: [-1, 1] \times D^{l-1} \to D^l$ be a combinatorial equivalence defined by radial shrinking.

$s_3: D^l \approx$ fiber over $p \in P$ in a tubular neighborhood of P in Q.

Then f can be described as $(x, y, z) \mapsto s_3 \circ s_2 \circ s_1(-2|x| + 2, z)$, where $|x|$ denotes the length of $x \in D_2^{k+1}$.

LEMMA 1.6. *There is a map r from the square $0 \leqq x, y \leqq 2$ in the plane to R which is differentiable and transverse regular to $0 \in R$ and such that $r(2, y) = -2$ for all y, $r(x, 2) = -2x + 2$, and such that $r^{-1}(0)$ is a connected 1-dimensional submanifold of the square containing the segments $\{0 \leqq x \leqq 1/4, y = 1/4\}$, $\{x = 1, 1 \leqq y \leqq 2\}$ and such that*

$$r^{-1}(0) \cap (\{1 \leqq y \leqq 2\} \cup \{1 \leqq x \leqq 2\} \cup \{0 \leqq y \leqq 1/4\})$$

is just these segments. Further, one can choose such a function r so that $r(x, y) = -2$ for $y \leqq 1/10$, and $r(x, y) \geqq 1.1$ for $(x, y) \in \{x \leqq 1/10, y \geqq 1.8\}$, and so that r is a function of y only on the set $\{x \leqq 1/10, y \leqq 1.8\}$.

Proof. Define a function in polar coordinates in this region of the plane by $g(r, \theta) = r^2 \cdot (\cos^2(\theta - \pi/4) - \sin^2(\theta - \pi/4)) + c(r^2)$ where $c: [0, 4] \to R$ is a C^∞ decreasing map with $c(0) = 1$, $c(\frac{1}{2}) = \frac{1}{2}$, and $c(\tau) = 0$ for $\tau \geqq \frac{3}{4}$. Let $h_1(x, y)$ be this same function in ordinary orthogonal coordinates.

Let $h_2(x, y) = -h_1(x-1, y-1/4)$, and $h_3(x, y) = p(x) \cdot h_2(x, y) + p(x) - 1$, where $p(x)$ is C^∞, decreasing, and $p(x) = 1$ for $x \leqq 1.1$, $p(x) = 0$ for $x \geqq 1.25$.

Let $h_4(x, y) = h_3(x, y) + q(x)$ where $q(x)$ is C^∞, decreasing, and $q(x) = 0$ for

$x \leqq 1.5$, $q(2) = -1$, and then define $r(x, y)$ to be $r(x, y) = s(y) \cdot (h_4(x, y)) + (1 - s(y)) \cdot (-2x + 2)$, where s is a C^∞ decreasing map, $s(y) = 1$ for $y \leqq 1.5$, $s(y) = 0$ for $y \geqq 1.75$. This function r satisfies the conditions of the first sentence of the lemma, and clearly may be further modified to satisfy the conditions of the second sentence also.

Now define a homotopy

$$F: D_2^{k+1} \times D_2^r \times D^{l-1} \times [0, 1] \to R \times D^{l-1}$$

by

$$F(x, y, z, \tau) = ((1-\tau) \cdot (-2|x| + 2) + \tau \cdot r(|x|, |y|), z).$$

Then $s_3 \circ s_2 \circ s_1 \circ F$ defines a homotopy of f restricted to the tubular neighborhood of $h(D_2^{k+1})$ in our coordinates, which is stationary on the boundary of this tubular neighborhood. Thus we may think of $s_3 \circ s_2 \circ s_1 \circ F$ as defining a homotopy of $f: M \to Q$ to a new map which we denote $f_1: M \to Q$, since

$$s_3 \circ s_2 \circ s_1 \circ F(x, y, z, 0) = s_3 \circ s_2 \circ s_1(-2|x| + 2, z) = f(x, y, z).$$

Note that $s_3 \circ s_2 \circ s_1 \circ F(x, y, z, 1) = s_3 \circ s_2 \circ s_1(r(|x|, |y|), z)$ is differentiable in a neighborhood of the universe image of $p \in P$ and is transverse regular over $p \in P$, with the inverse image of p diffeomorphic to

$$\begin{aligned} D^{k+1} \times S^{r-1} &= D^{k+1} \times S^{m-l-k-1} && \text{in Case (a)} \\ &= D^{k+1} \times S^{m-l-k} && \text{in Case (b).} \end{aligned}$$

Thus we end with a pair (M, f_1), where $f_1: M \to Q$ is differentiable and transverse regular over $P \subset Q$ and $f_1^{-1}(P)$ is just the manifold $N = f^{-1}(P)$ with a surgery done on the class of $i(S^k)$. As noted before, the map f_1 is tangential, so we have regained our original assumptions and can proceed inductively.

Now consider the separate cases:

Case (a): It is clear from the work of Novikov [7], [12] that one can proceed inductively through a series of surgeries as above to arrive at a pair $(M, \bar{f})$, where $\bar{f}$ is differentiable and transverse to $P \subset Q$ and $\bar{f}|\bar{f}^{-1}(P): \bar{f}^{-1}(P) \to P$ induces isomorphisms of homology for all dimensions less than or equal to $(m-l)/2 - 1$, and $\bar{f}^{-1}(P)$ simply connected. That is, surgeries as above can be used to kill the kernel of $H_k(f^{-1}(P)) \to H_k(P)$ for $k \leqq (m-l)/2 - 1$. The cases $m - l \equiv 0, 1, 3 \bmod 4$ are distinct.

If $m - l \equiv 0 \bmod 4$, then ordinary surgery may be done to the pair $(\bar{f}^{-1}(P), \bar{f}|\bar{f}^{-1}(P))$ until homology isomorphism results in the middle dimension also. (A proof of this may be found in [7, pp. 26–28], or [12, pp. 288–290]. One must choose the homology classes in $\ker f_*$ to be killed with care.) It is further clear that these surgeries can be done ambiently as above, since the only obstruction to this is that we make a specific choice of framing of $\nu(i(S^k) \subset N)$ in Lemma 1.5, and the proof that surgery can be used to kill the kernel of f_* in dimension $(m-l)/2$ is independent of the specific framing used in performing the surgery. Hence, for $m - l \equiv 0 \bmod 4$, surgery may be completed in the middle dimension, and we end with a map

$f': M \to Q$ transverse to $P \subset Q$ such that $f'|f'^{-1}(P): f'^{-1}(P) \to P$ is a tangential homotopy equivalence as required in Theorem 1.

For $m-l \equiv 1 \bmod 4$ the same argument applies. Ordinary surgery may be done to $(\bar{f}^{-1}(P), \bar{f}|\bar{f}^{-1}(P))$ until homology isomorphism results in dimension $(m-l-1)/2$, as proven in [12, pp. 294–302]. Again one sees these surgeries can all be done ambiently in M as above, since the proof is independent of the choice of frame on $\nu(i(S^k) \subset N)$. Thus we again end with a map $f': M \to Q$ as required by Theorem 1.

For $m-l \equiv 3 \bmod 4$, we must use the note following Lemma 1.5. Novikov proves that surgery can modify $(\bar{f}^{-1}(P), \bar{f}|\bar{f}^{-1}(P))$ so that homology isomorphisms in dimension $(m-l-1)/2$ result, but this proof requires use of specific framings on $\nu(i(S^{(m-l-1)/2}) \subset N)$ in performing the surgeries. The freedom to vary a given framing by any stably trivial element of $\pi_{(m-l-1)/2}(SO(m-l+1)/2))$ is, however, all that is needed. (See [12, pp. 298–302].) Since we may use these frames in doing ambient surgery, we again end with a map $f': M \to Q$ as required in Theorem 1, and Theorem 1 is proven.

Case (b). Recall in this case we have $f: M^{m+1} \to Q$ a tangential homotopy equivalence, where M^{m+1} is an h-cobordism between M_1^m and M_2^m, and we are doing surgery to $N = f^{-1}(P)$. Note first that since we are doing surgeries on classes represented by imbeddings $i: S^k \to N$ where $i(S^k) \cap \partial M = \varnothing$, the homotopies of f we define may be assumed stationary on ∂M.

As in Novikov [12], [7, Proposition 4, p. 19] we get a sequence of modifications as above such that we arrive at a pair (M^{m+1}, f_1) where f_1 is differentiable and transverse to P, and $f_{1*}: H_*(f_1^{-1}(P)) \to H_*(P)$ is an isomorphism in dimensions less than or equal to $(m-l)/2-1$, and $f_1^{-1}(P)$ simply connected. $f_1^{-1}(P)$ has dimension $m+1-l$, where $m-l \equiv 0 \bmod 4$, so we may refer to [7, pp. 28–35], [12, p. 304] for a proof that ordinary surgery may be done to the pair $(f_1^{-1}(P), f_1|f_1^{-1}(P))$ until we get a pair with homology isomorphism in dimension $(m-l)/2$ as well, and thus get a homotopy equivalence.

Once again, the only obstruction to performing these surgeries ambiently as above is the choice of framing of $\nu(i(S^{(m-l)/2}) \subset f_1^{-1}(P))$ made in Lemma 1.5. Since the proof that surgeries can be chosen to kill the kernel of $f_{1*}: H_{(m-l)/2}(f_1^{-1}(P)) \to H_{(m-l)/2}(P)$ is independent of such choices, the surgeries may all be done in M^{m+1} as above.

Thus we can modify (M^{m+1}, f) to a pair (M^{m+1}, f') by a series of homotopies of f so that $f': M \to Q$ is a tangential homotopy equivalence transverse to $P \subset Q$, $f|\partial M = f'|\partial M$, and if $N = f'^{-1}(P)$, then $f'|N: N \to P$ is a homotopy equivalence, so that N is an h-cobordism between N_1 and N_2. Thus Theorem 2 is proven.

Note 1. The reader should notice that the following weaker analogue of the uniqueness theorem can be proven for the cases $m-l \equiv 1$ or $3 \bmod 4$, using the analogue of Case (b) above:

Let P and Q be as in Theorem 1, with $m-l \equiv 1$ or $3 \bmod 4$. Assume M^{m+1}, $f: M^{m+1} \to Q$, M_i and N_i for $i = 1, 2$ are all as in Theorem 2, with $f|N_i: N_i \to P$ a homotopy equivalence for $i=1, 2$. Then f is homotopic via a homotopy which is stationary on ∂M^{m+1} to a map $f'\colon M^{m+1} \to Q$ such that $\pi_1(f'^{-1}(P))=0$ and $f'_*\colon H_i(f'^{-1}(P)) \to H_i(P)$ is an isomorphism for $i \leqq (m-l+1)/2-1$. As in [12, pp. 304–306], N_1 is diffeomorphic to $N_2 \# \Sigma^{m-l}$ for some homotopy sphere Σ^{m-l} which bounds a parallelizable manifold.

Note 2. Before proceeding we should note that the following stronger analogue of Theorem 1 holds for manifolds with boundary.

THEOREM. *Suppose Q^m and P^{m-l} and M^m are manifolds with boundary with P^{m-l} a submanifold of Q^m so that $\partial P \subset \partial Q$, where $l \geqq 3$, $m-l>6$, $m-l \neq 14$, and $\pi_1(Q)=\pi_1(\partial Q)=\pi_1(P)=\pi_1(\partial P)=0$. Suppose $f\colon (M^m, \partial M^m) \to (Q^m, \partial Q^m)$ is a degree $+1$ tangential homotopy equivalence. Then f is homotopic to a map $f'\colon (M^m, \partial M^m) \to (Q^m, \partial Q^m)$ which is differentiable and transverse regular over $(P, \partial P) \subset (Q, \partial Q)$ and such that $f'|f'^{-1}(P, \partial P)\colon f'^{-1}(P, \partial P) \to (P, \partial P)$ is a homotopy equivalence.*

The proof of this is similar in spirit to the work above. One readily shows the analogues of Lemma 1.1 and Proposition 1.1. Ambient surgery is then carried out on $f^{-1}(\partial P)$ and $f^{-1}(P)$ as above up to the middle dimension. The obstructions to middle dimensional surgery can be made to vanish since we allow our homotopies to alter f on ∂M^m. The interested reader will find a discussion and proof of this, including the necessary algebra and the techniques needed to avoid middle dimension obstructions, in Wagoner [17] and Wall [18].

2. **Homotopy quaternionic projective spaces.** We refer the reader to the introduction for the definition of $\theta(QP_n)$, which is defined for $n \geqq 2$.

LEMMA 2.1. *The equivalence relation defining $\theta(QP_n)$ can be described as follows: $(M,f) \sim (N, g)$ if and only if M and N are diffeomorphic by a map $d\colon M \to N$ such that f is homotopic to $g \circ d$.*

Proof. $(M,f) \sim (N, g)$ in $\theta(QP_n)$ implies there is an h-cobordism C and a tangential homotopy equivalence $h\colon C \to QP_n$. Since dimension M, N is greater than or equal to 5, there is a diffeomorphism $d\colon M \times I \to C$ with $d(x, 0)=x$. Hence $d|(M \times (1))$ is a diffeomorphism $M \to N$ such that $h \circ d|(M \times (1))$ is homotopic to $h \circ d|(M \times (0))$. Since h restricts to f and g, $f \circ d$ is homotopic to $g \circ d$, and so f is homotopic to $g \circ d$.

Conversely, if (N, g) represents a class in $\theta(QP_n)$ then $g \circ$ projection: $N \times I \to QP_n$ is a tangential homotopy equivalence. Assume $d\colon M \to N$ is a diffeomorphism such that $g \circ d$ is homotopic to $f\colon M \to QP_n$. Identify $N \times (1) \subset N \times I$ with M via d to make $N \times I$ an h-cobordism of N and M. Then $h = g \circ$ projection restricts to g on $N \times (0)$ and to a map homotopic to f on $M = N \times (1)$, and may be modified in a tubular neighborhood of M in $N \times I$ to restrict to f.

LEMMA 2.2. *Suppose $f\colon M^{4n} \to QP_n$ is a tangential homotopy equivalence and Σ^{4n} represents an element of θ_{4n}. Let $i\colon D^{4n} \to M$ be an imbedding. Then for some* *diffeomorphism of boundaries* $d\colon \partial D^{4n} \to \partial(M - i(D^{\circ 4n}))$, *the connected sum* $M \# \Sigma^{4n}$ *is diffeomorphic to* $(M - i(D^{\circ 4n})) \cup_d D^{4n}$. *Define* $\mathrm{id}'\colon M \# \Sigma^{4n} \to M$ *by* $\mathrm{id}'(x) = x$ *for* $x \in M - i(D^{\circ 4n})$, *and* $\mathrm{id}'(s, \tau) = i(i^{-1}(d(s)), \tau)$ *for* $(s, \tau) \in D^{4n}$, *where we view* D^{4n} *as* $\{(s, \tau) | s \in S^{4n-1}, \tau \in I\}$ *modulo* $(s, 0) \sim (r, 0)$ *for any* $s, r \in S^{4n-1}$. *Then* id' *is a tangential map, and thus* $f \circ \mathrm{id}'\colon M \# \Sigma^{4n} \to QP_n$ *is a tangential homotopy equivalence.*

Proof. The existence of the required diffeomorphism $d\colon \partial D^{4n} \to \partial(M - i(D^{\circ 4n}))$ is proven by Smale [13], so all that remains is to show id' is tangential.

$\mathrm{id}'|(M - i(D^{\circ 4n}))$ is identity, so we may collapse $(M - i(D^{\circ 4n}))$ to a point to get a diagram with stable bundles

$$\begin{array}{ccc} \tau(M \# \Sigma^{4n}) - \mathrm{id}'^*(\tau(M)) & \xrightarrow{\approx} & \eta \\ \downarrow & & \downarrow \\ M \# \Sigma^{4n} & \longrightarrow & S^{4n} = D^{4n}/\partial D^{4n} \end{array}$$

where the diagram defines the stable bundle η. We need only show η is trivial to prove id' is tangential.

Since a stable bundle over S^{4n} is trivial if and only if its nth Pontrjagin class is zero, we investigate the nth Pontrjagin class of $\tau(M \# \Sigma^{4n}) - \mathrm{id}'^*(\tau(M))$ which is zero if and only if the nth Pontrjagin class of η is. In the Mayer-Vietoris cohomology sequence of the triad $(M \# \Sigma^{4n}; M - i(D^{\circ 4n}), D^{4n})$ inclusion: $M - i(D^{\circ 4n}) \to M \# \Sigma^{4n}$ induces isomorphism of cohomology for all dimensions less than $4n-2$. Let $p_1, \ldots, p_n$ denote the Pontrjagin classes of M, and $p_1', \ldots, p_n'$ the Pontrjagin classes of $M \# \Sigma^{4n}$. Since $\mathrm{id}'|(M - i(D^{\circ 4n}))$ is tangential, $p_1' = \mathrm{id}'^*(p_1), \ldots, p_{n-1}' = \mathrm{id}'^*(p_{n-1})$. index $(M \# \Sigma^{4n}) = $ index (M), so if L_n is the nth Hirzebruch polynomial, then

$$\begin{aligned} L_n(p_1', \ldots, p_n')[M \# \Sigma^{4n}] &= L_n(p_1, \ldots, p_n)[M] \\ &= L_n(\mathrm{id}'^*(p_1), \ldots, \mathrm{id}'^*(p_n))[M \# \Sigma^{4n}] \\ &= L_n(p_1', \ldots, p_{n-1}', \mathrm{id}'^*(p_n))[M \# \Sigma^{4n}], \end{aligned}$$

where we orient $M \# \Sigma^{4n}$ so that id' is a degree $+1$ map.

If $L_n(X_1, \ldots, X_n)$ is the nth Hirzebruch polynomial, the coefficient of X_n is nonzero. Thus we must have $p_n' = \mathrm{id}'^*(p_n)$ above, and hence the nth Pontrjagin class of $\tau(M \# \Sigma^{4n}) - \mathrm{id}'^*(\tau(M))$ is zero and η is trivial as needed.

Definition. θ_{4n} acts on $\theta(QP_n)$ in the following manner: $[\Sigma^{4n}] \cdot [(M, f)] \equiv [(M \# \Sigma^{4n}, f \circ \mathrm{id}')]$. This action is well defined by Lemma 2.2 of Kervaire-Milnor [6].

Definition. Define $r\colon \theta(QP_n) \to \theta(QP_{n-1})$ for all $n \geqq 3$ as follows: Given $\alpha \in \theta(QP_n)$, we choose a representative element (M, f). QP_n has dimension $4n$, and codimension of QP_{n-1} in QP_n is 4, QP_n is simply connected for all n, so by Theorem 1 and the fact that we are free within the class α to deform f by arbitrary homotopies, we may assume (M, f) is chosen with f differentiable and transverse to

$QP_{n-1} \subset QP_n$, and such that $f|f^{-1}(QP_{n-1})\colon f^{-1}(QP_{n-1}) \to QP_{n-1}$ is a tangential homotopy equivalence. Define $r(\alpha)$ to be the class of $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$ in $\theta(QP_{n-1})$. To show that the image of α depends only on α, suppose $(N, g) \in \alpha$ is such that $g\colon N \to QP_n$ is transverse to $QP_{n-1} \subset QP_n$, and $g|g^{-1}(QP_{n-1})\colon g^{-1}(QP_{n-1}) \to QP_{n-1}$ is a homotopy equivalence. Let (C, h) be a cobordism between (M, f) and (N, g) as in the definition of equivalence in $\theta(QP_n)$. By Theorem 2, we may assume $h\colon C \to QP_n$ is transverse regular over $QP_{n-1} \subset QP_n$, and that $h|h^{-1}(QP_{n-1})\colon h^{-1}(QP_{n-1}) \to QP_{n-1}$ is a tangential homotopy equivalence. Then $(h^{-1}(QP_{n-1}), h|h^{-1}(QP_{n-1}))$ is a cobordism between $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$ and $(g^{-1}(QP_{n-1}), g|g^{-1}(QP_{n-1}))$, and so these represent the same class in $\theta(QP_{n-1})$.

LEMMA 2.3. *Suppose (M, f_1) represents $\alpha \in \theta(QP_n)$, with $n \geqq 3$. Then f_1 is homotopic to $f\colon M \to QP_n$ such that*

(a) *f is differentiable and transverse regular over $QP_{n-1} \subset QP_n$, and*

$$(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$$

is a representative of $r(\alpha)$.

(b) *If we consider QP_n as the Thom space of $\nu(QP_{n-1} \subset QP_n)$, then f defines a bundle isomorphism $\nu(f^{-1}(QP_{n-1}) \subset M) \approx \nu(QP_{n-1} \subset QP_n)$ on a tubular neighborhood ν of $f^{-1}(QP_{n-1})$, and f maps the complement of ν to the distinguished point in $QP_n = T(\nu(QP_{n-1} \subset QP_n))$.*

Proof. See Thom [16] and the definition of $r(\alpha)$.

LEMMA 2.4. *For any $\alpha \in \theta(QP_n)$ with $n \geqq 3$, if a representative pair (M, f) for α is chosen as in Lemma 2.3 then the tubular neighborhood ν of $f^{-1}(QP_{n-1})$ in M is bounded by a sphere, and M is diffeomorphic to $\nu \cup_i D^{4n}$ for some attaching diffeomorphism of boundaries $i\colon \partial D^{4n} \to \partial\nu$.*

Proof. There are disc bundle isomorphisms $\nu \approx f^*(\nu(QP_{n-1} \subset QP_n)) \approx f^*(\mathscr{S}^4)$, where $\mathscr{S}^4$ is the 4-disc bundle over QP_{n-1} described in the introduction. Hence there is a 3-sphere bundle isomorphism $\partial\nu \approx f^*(\mathscr{S}^3_0)$, where $\mathscr{S}^3_0$ has total space S^{4n-1}. Then naturality of the homotopy sequence of a bundle implies $\partial\nu$ has the homotopy type of S^{4n-1}.

Let $D = \partial\nu \cup$ (complement of ν in M), so that $\nu \cup D = M$, $\nu \cap D = \partial\nu$. From the Mayer-Vietoris sequence of this decomposition of M we see inclusion induces homology isomorphisms $H_i(\nu) \approx H_i(M)$ for $i \leqq 4n-1$, and so $H_i(D) = 0$ for $i \leqq 4n-2$. By duality, $H_{4n}(D) \approx H^0(D, \partial\nu) = 0$ and $H_{4n-1}(D) \approx H'(D, \partial\nu) = 0$, so $H_i(D) = 0$ for all $i \leqq 4n$.

Applying Van Kampen's Theorem to M with subsets $\nu^\circ =$ interior of ν, and $M - f^{-1}(QP_{n-1})$, we see D is simply connected, since D has the homotopy type of $M - f^{-1}(QP_{n-1})$. Hence D is contractible, and $\partial D = \partial\nu$ is diffeomorphic to S^{4n-1}, as needed. See [13].

DEFINITION. Define $h\colon \theta(QP_{n-1}) \to \theta_{4n-1}$ for $n \geqq 3$ as follows. Given $\beta \in \theta(QP_{n-1})$ choose a representative (P, t) in which $t\colon P \to QP_{n-1}$ is differentiable. As in the proof of Lemma 2.4, $t^*(\mathscr{S}_0^3)$ has the homotopy type of S^{4n-1}. We define $h(\beta)$ to be the class in θ_{4n-1} represented by $t^*(\mathscr{S}_0^3)$, and check that h is now a well defined map.

If $(P, t)\sim(R, s)$ in $\theta(QP_{n-1})$, where both s and t are differentiable maps, we may choose a cobordism between them (C', h') such that $h'\colon C' \to QP_{n-1}$ is differentiable. Then $h'^*(\mathscr{S}_0^3)$ is a manifold of the homotopy type of S^{4n-1}, and is thus an h-cobordism between $t^*(\mathscr{S}_0^3)$ and $s^*(\mathscr{S}_0^3)$.

We now proceed with the proof of Theorem 3. Given $\alpha \in \theta(QP_n)$, choose a representative (M,f) for α as in Lemma 2.3 so that $[(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))]$ $=r(\alpha)$, and $f\colon M \to QP_n$ is transverse to QP_{n-1}. Then $(f|f^{-1}(QP_{n-1}))^*(\mathscr{S}_0^3)$ $=\partial(\nu(f^{-1}(QP_{n-1})\subset M))\approx S^{4n-1}$ by Lemma 2.4, and thus image of r is contained in the kernel of h.

Suppose $\beta \in \theta(QP_{n-1})$ and $h(\beta)=[S^{4n-1}] \in \theta_{4n-1}$. Take $(R, s) \in \beta$ with $s\colon R \to QP_{n-1}$ differentiable, and identify the boundary of $s^*(\mathscr{S}^4)$ with S^{4n-1}. In this fashion we attach D^{4n} to $s^*(\mathscr{S}^4)$ differentiably. Let $R_1 = s^*(\mathscr{S}^4) \cup D^{4n}$ be the result of such an attachment. Considering QP_n as $\nu(QP_{n-1}\subset QP_n) \cup D^{4n}$, or $\mathscr{S}^4 \cup D^{4n}$, by identifying a tubular neighborhood of QP_{n-1} with $\mathscr{S}^4$, we get a homotopy equivalence $s_1\colon R_1 \to QP_n$ by radial extension of the bundle map $s^*(\mathscr{S}^4) \to \mathscr{S}^4$. By a proof entirely similar to that of Lemma 2.2, one can show that s_1 is a tangential map, and thus that (R_1, s_1) represents an element γ of $\theta(QP_n)$. Clearly $r(\gamma)=[(R, s)]$ $=\beta \in \theta(QP_{n-1})$, and $\beta \in$ image (r). Thus we have shown image $(r)=$kernel (h).

Suppose now $\alpha, \beta \in \theta(QP_n)$ are such that $r(\alpha)=r(\beta)$. Choose representative elements (M,f) and (N, g) for α and β respectively as in Lemma 2.3. Then there is a cobordism between $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$ and $(g^{-1}(QP_{n-1}), g|g^{-1}(QP_{n-1}))$ since $r(\alpha)=r(\beta)$, and, since f and g are each differentiable in the respective inverse images of QP_{n-1}, we may assume this cobordism is (C, h) with $h\colon C \to QP_{n-1}$ differentiable.

Associated with the resulting manifold $h^*(\mathscr{S}^4)$, a differentiable disc bundle, is the boundary 3-sphere bundle $h^*(\mathscr{S}_0^3)$. As before, naturality of the homotopy sequence of a bundle shows that $h^*(\mathscr{S}_0^3)$ has the homotopy type of S^{4n-1}, and thus that $h^*(\mathscr{S}_0^3)$ is an h-cobordism between

$$(f|f^{-1}(QP_{n-1}))^*(\mathscr{S}_0^3) \quad \text{and} \quad (g|g^{-1}(QP_{n-1}))^*(\mathscr{S}_0^3).$$

Since $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1})) \in r(\alpha)$, and image $(r)=$kernel (h), $h^*(\mathscr{S}_0^3)$ is thus diffeomorphic to $S^{4n-1} \times I$.

Attach $D^{4n} \times I$ to $h^*(\mathscr{S}^4)$ by a diffeomorphism $S^{4n-1} \times I \approx \partial h^*(\mathscr{S}^4)$, thus creating a manifold C', and a homotopy equivalence $h'\colon C' \to QP_n$, the radial extension of the natural bundle map. Then C' is an h-cobordism between its boundary components, which are respectively $M \# \Sigma^{4n}$ and $M \# \Sigma'^{4n}$, where Σ^{4n} and Σ'^{4n} are homotopy $4n$-spheres. We now check that $h'\colon C' \to QP_n$ is tangential, and thus that (C', h') is a cobordism between $[\Sigma^{4n}]\cdot\alpha$ and $[\Sigma'^{4n}]\cdot\beta$ in $\theta(QP_n)$.

The proof that h' is tangential is as in Lemma 2.2. If $i\colon h^*(\mathscr{S}^4) \to C'$ is inclusion, then $i^*(\tau(C'))\approx i^*h'^*(\tau(QP_n))$ and hence the first $n-1$ Pontrjagin classes of $\tau(C')$

and $h'^*(\tau(QP_n))$ are the same. Since $\tau(C')$ and $h'^*(\tau(QP_n))$ are stably isomorphic over $h^*(\mathcal{S}^4)\subset C'$, we may collapse this subspace to a line, and by analogy with Lemma 2.2 define a bundle η over $S^{4n}\times I$. One can then show $\eta|S^{4n}\times(0)$ is stably trivial by computing its nth Pontrjagin class, and thus prove that $h'\colon C'\to QP_n$ is tangential. Then $[\Sigma^{4n}]\cdot\alpha$ and $[\Sigma'^{4n}]\cdot\beta$ represent the same class in $\theta(QP_n)$, and α and β differ by an action of θ_{4n} in $\theta(QP_n)$.

Conversely, choosing representatives as in Lemma 2.3, it is clear that if α and β differ by some action of θ_{4n}, then $r(\alpha)=r(\beta)$. This completes the proof of Theorem 3.

Construction. Theorem 3 gives us an inductive procedure whereby we can construct representatives of all elements of $\theta(QP_n)$ from elements of $\theta(QP_{n-1})$ for $n\geqq 3$.

Given any representative (M,f) of $\alpha\in\theta(QP_n)$, we may deform f by a homotopy and assume it is in the form prescribed by Lemma 2.3. Then $r(\alpha)$ is represented by $(f^{-1}(QP_{n-1}), f|f^{-1}(QP_{n-1}))$, and $(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}^3_0)$ is diffeomorphic to S^{4n-1}. Attach D^{4n} to $(f|f^{-1}(QP_{n-1}))^*(\mathcal{S}^4)$ by a diffeomorphism of boundaries, thus creating a new manifold M' and a homotopy equivalence $f'\colon M'\to QP_n$, which we have seen in the proof of Theorem 3 is tangential. Then (M',f') represents an element of $\theta(QP_n)$, and $r([(M',f')])=r(\alpha)$, and so by Theorem 3 $[(M,f)]$ and $[(M',f')]$ differ by some action of θ_{4n} in $\theta(QP_n)$. Thus we see all tangential homotopy QP_n's arise from tangential QP_{n-1}'s by the operation of pulling back the bundle $\mathcal{S}^4$, and capping the result with a disc D^{4n}.

3. $\theta(QP_2)$.

The existence of the inductive procedure for constructing tangential homotopy quaternionic projective spaces given in the last section leads one to investigate the starting point, $\theta(QP_2)$.

Suppose (M,f) represents an element of $\theta(QP_2)$. Let $i\colon S^4\to M^8$ be an imbedding representing a generator of $\pi_4(M^8)\approx\pi_4(QP_2)\approx Z$.

LEMMA 3.1. *Let ν denote a tubular neighborhood of $i(S^4)$ in M. Then $D=M$-interior(ν) is diffeomorphic to the 8-disc, D^8. Consequently, the 3-sphere bundle which bounds $\nu(i(S^4)\subset M)$ has total space diffeomorphic to S^7.*

Proof. Since M is simply connected and D is a deformation retract of $M-i(S^4)$, D is also simply connected. For $q<8$, $H^q(M)\approx H^q(i(S^4))$, so $H_j(D)\approx H^{8-j}(M, i(S^4))$ $=0$ for $j\neq 0$. Hence D is contractible, and $D\approx D^8$ by the h-cobordism theorem.

We now investigate $\nu(i(S^4)\subset M)=\nu$. Since $f\colon M\to QP_2$ is tangential, $\nu\oplus\tau(S^4)$ $\approx i^*(\tau(M))\approx i^*f^*(\nu(QP_1\subset QP_2)\oplus\tau(S^4))$, where the isomorphisms indicated are stable isomorphisms, and thus ν is stably isomorphic to $i^*f^*(\nu(QP_1\subset QP_2))$.

Represent 3-sphere bundles over S^4 by elements of $\pi_3(SO(4))$ as in Steenrod [14, §§18–22], using the isomorphism described by Milnor [9], $\pi_3(SO(4))\approx Z\oplus Z$ under the correspondence $(h,j)\Leftrightarrow f_{hj}$, where $f_{hj}\colon S^3\to SO(4)$ is defined by $f_{hj}(u)\cdot v=u^h\cdot v\cdot u^j$ for $v\in R^4$. (Quaternion multiplication is indicated on the right.) It is know that under this isomorphism $\nu(QP_1\subset QP_2)$ corresponds to $(1,0)\in Z\oplus Z$. See [9].

LEMMA 3.2. *If $\mathscr{S}_{hj}$ is the 3-sphere bundle over S^4 corresponding to (h, j) in this representation, then the first Pontrjagin class of $\mathscr{S}_{hj}$ is $p_1(\mathscr{S}_{hj}) = \pm 2(h-j)i$, where $i \in H^4(S^4)$ is a generator.*

Proof. See Milnor [9].

LEMMA 3.3. *$\nu = \nu(i(S^4) \subset M)$ is represented by an element of the form*

$$(n+1, n) \in Z \oplus Z.$$

Proof. $p_1(\tau(S^4)) = 0$, so by Lemma 3.2 $\tau(S^4)$ is represented by an element of $Z \oplus Z$ of the form (s, s). Kernel $\pi_3(SO(4) \to \pi_3(SO)) = Z$, and since the element corresponding to $\tau(S^4)$ is in this kernel, the kernel is contained in the diagonal of $Z \oplus Z$. Thus ν is represented by $(1, 0) + (n, n) = (n+1, n)$ for some $n \in Z$ since $\nu(QP_1 \subset QP_2)$ is represented by $(1, 0)$ and is stably isomorphic to ν.

LEMMA 3.4. *$\nu = \nu(i(S^4) \subset M) \approx \nu(QP_1 \subset QP_2)$.*

Proof. We will show that among those 3-sphere bundles over S^4 with representatives of the form $(n+1, n)$ in $\pi_3(SO(4)) \approx Z \oplus Z$, only the one corresponding to $(1, 0)$ has a total space of the homotopy type of S^7.

Let T be the total space of some 4-disc bundle over S^4 with associated 3-sphere bundle denoted by ∂. $\pi_1(\partial) = \pi_2(\partial) = 0$, then, and the first possibly nonvanishing homology group of ∂ is $H_3(\partial)$. Consider the cohomology exact sequence

$$\begin{array}{ccccccccc}
H^3(\partial) & \to & H^4(T, \partial) & \xrightarrow{j^*} & H^4(T) & \to & H^4(\partial) & \to & H^5(T, \partial) \\
 & & \uparrow \phi & \nearrow & & & & & \uparrow \phi \\
 & & H^0(S^4) & & & & & & H^1(S^4) = 0
\end{array}$$

where the maps denoted by ϕ are the Thom isomorphisms. The image of a generator of $H^0(S^4)$ in $H^4(T)$ under $j^* \circ \phi$ is a generator if and only if $H^4(\partial) = 0$, and it is known that the Euler class of the 4-disc bundle of which T is the total space is the image of a generator $H^0(S^4)$ under $j^* \circ \phi$.

The process of taking Euler classes of corresponding bundles gives a homomorphism $\pi_3(SO(4)) \to H^4(S^4)$ (see [8]), and the Euler class of $\tau(S^4)$ is twice a generator of $H^4(S^4)$, so in the diagonal set $\{(n, n) \mid n \in Z\}$ of $Z \times Z$, all bundles represented have nonzero Euler classes except for $(0, 0)$. The Euler class of the bundle corresponding to $(1, 0)$ is a generator of $H^4(S^4)$ since this bundle is just $\partial\nu(QP_1 \subset QP_2)$ and has total space S^7. Thus $(1, 0)$ is the only element of $Z \oplus Z$ of the form $(n+1, n)$ whose corresponding bundle has Euler class a generator of $H^4(S^4)$, and $(1, 0)$ is the only element of the form $(n+1, n)$ representing a 3-sphere bundle whose total space has the homotopy type of S^7. Since the total space of ν is S^7, we see $\nu \approx \nu(QP_1 \subset QP_2)$.

So, any manifold M^8 of the tangential homotopy type of QP_2 is of the form $M=T\cup D^8$, where T is the total space of the differentiable disc bundle $\nu(QP_1\subset QP_2)$, and D^8 is attached to T by a diffeomorphism of boundaries. Thus we have proven:

PROPOSITION 3.1. *Any 8 manifold without boundary of the tangential homotopy type of QP_2 is diffeomorphic to a connected sum $QP_2 \# \Sigma^8$, where Σ^8 is a homotopy 8-sphere. Thus, there are at most two distinct such manifolds, and these are combinatorially equivalent* [11].

To finish the computation of $\theta(QP_2)$, we investigate the homotopy classes of maps $f: M \to QP_2$ which contain tangential homotopy equivalences. If $f: M\to QP_2$ and $g: M \to QP_2$ are two such equivalences, and if f and g are not homotopic, then $f\circ g^{-1}: QP_2 \to QP_2$ is a tangential homotopy equivalence which is not homotopic to identity.

The remainder of this section is devoted to a proof that there is only one homotopy class of maps from QP_2 to QP_2 which contains a tangential homotopy equivalence. This will finish the proof of Theorem 4.

Let $f: QP_2 \to QP_2$ be a tangential homotopy equivalence. It is known [3] that $p_1(QP_2)=$ 1st Pontrjagin class of QP_2 is twice a generator in $H^4(QP_2)\approx Z$, and thus tangentiality of f implies $f_*: \pi_4(QP_2)\to \pi_4(QP_2)$ is identity. So, in the standard C-W complex structure of QP_2 we may assume f is the identity map on $S^4=$4-skeleton of QP_2, and that $h_\tau: S^4 \to S^4$, the identity map for all $\tau\in I$, is a homotopy between f and identity on the 7-skeleton of QP_2, which is the same as the 4-skeleton.

The partial homotopy of f with identity given by h_τ gives rise to an obstruction cohomology class $\delta^8(f, \text{identity}, h_\tau)\in H^8(QP_2, \pi_8(QP_2))\approx Z_2$, since the homotopy sequence of the bundle $S^3\to S^{11}\to QP_2$ shows $\pi_8(QP_2)\approx \pi_7(S^3)\approx Z_2$. If this obstruction class is zero then the partial homotopy h_τ may be extended over QP_2 to give a homotopy of f with identity. Moreover, if g is another such map, and $\delta^8(f, \text{id}, h_\tau)\neq 0$, $\delta^8(g, \text{id}, h_\tau)\neq 0$, then f and g are homotopic by the additivity of obstruction cochains. Thus, we may finish the proof of the theorem by constructing a map $t: QP_2\to QP_2$ which is identity on the 4-skeleton of QP_2 and which is homotopic to identity: $QP_2 \to QP_2$, but whose obstruction class $\delta^8(t, \text{id}, h_\tau)$ is not zero. (The homotopy of t to identity cannot then be stationary on the 4-skeleton of QP_2.)

We view QP_2 as the Thom space of the 4-disc bundle $\mathscr{S}^4$ over $S^4\approx QP_1$, and represent $\mathscr{S}^4$ as follows. $\mathscr{S}^4$ can be thought of as $\{(s, v) \mid s \text{ a quaternionic line in } Q^2, v \text{ a point in } s\}$. Define a map from $\{(a, b, q) \mid a, b, q \in Q;\ |a|^2+|b|^2=1,\ |q|\leqq 1\}$ to $\mathscr{S}^4$ taking (a, b, q) to (the line containing (a, b), $q\cdot(a, b)$). If (a, b, q) and (a', b', q') have the same image under this map then $(a, b)=s\cdot(a', b')$ for some $s\in Q$, $|s|=1$, and $q\cdot s\cdot(a', b')=q'\cdot(a', b')$ so that $q=q'\cdot s^{-1}$. Thus we can represent $\mathscr{S}^4$ over S^4 as $\{(a, b, q) \mid a, b, q\in Q,\ |a|^2+|b|^2=1,\ |q|\leqq 1\}$ modulo the equivalence relation $(a, b, q)\sim(a', b', q')$ if and only if $a=s\cdot a'$, $b=s\cdot b'$ and $q=q'\cdot s^{-1}$ for some $s\in Q$. $\mathscr{S}^3_0$ is given in this description by the set of equivalence classes which have a representative with third coordinate of length one. We view S^4 as $Q \cup \{\infty\}$, and identify $QP_1 = \{[(a, b, q)] \in \mathscr{S}^4 \mid q=0\}$ with S^4 under the correspondence $[(a, b, 0)] \Leftrightarrow a^{-1}b$, noting that if $(a, b, 0) = (q' \cdot c, q' \cdot d, 0)$ then $a^{-1}b = c^{-1}q'^{-1}q'd = c^{-1}d$.

Let $\eta: S^4 \to S^3$ be the essential map in $\pi_4(S^3) \approx Z_2$, where we think of S^3 as the unit length quaternions. Then, with the above identifications we define $t: QP_2 \to QP_2$ by $t([(a, b, q)]) = [(a, b, \eta(a^{-1}b) \cdot q)]$. One checks easily that this map $\mathscr{S}^4 \to \mathscr{S}^4$ is well defined on equivalence classes, and maps $\mathscr{S}^3_0$ to $\mathscr{S}^3_0$ and so does in fact give a map $QP_2 \to QP_2$. We wish to show that the obstruction to this map being homotopic to identity: $QP_2 \to QP_2$ via a homotopy which is stationary on the 4-skeleton of QP_2 is the nontrivial element in $\pi_8(QP_2)$.

Identify D^8 with $\{(a, b, c) \mid a, b \in Q, c \in R, |a|^2 + |b|^2 = 1, 0 \leqq c \leqq 1\}$ modulo the relation $(a, b, 0) \sim (c, d, 0)$ for any a, b, c, d. Then define $p: D^8 \to \mathscr{S}^4/\mathscr{S}^3_0 = QP_2$ by $p([(a, b, c)]) = [(a, b, 1-c^2)]$. Note that on the boundary of D^8 the map p is just the Hopf map $S^7 \to S^4$.

The obstruction class we wish to compute in $\pi_8(QP_2)$ is readily seen to be represented by the following map: View S^8 as two copies of D^8 with boundaries identified. On one hemisphere we define our map to be $p: D^8 \to QP_2$ as described above. On the other hemisphere we define our map to be $t \circ p: D^8 \to QP_2$ as above. Since t is identity on $S^4 = QP_1 \subset QP_2$ and p maps ∂D^8 to this S^4, this gives a continuous well-defined map $S^8 \to QP_2$ which is the Hopf map on the "equator" of S^8.

To show this map represents the nonzero class in $\pi_8(QP_2) \approx Z_2$, we will show the pull back of $\mathscr{S}^4$ by this map is a nontrivial bundle over S^8. Analogous to the above description of $\mathscr{S}^4$ over S^4, one may describe $\mathscr{S}^4$ over QP_2 as

$$\{(a, b, c, q) \mid a, b, c, q \in Q, |a|^2 + |b|^2 + |c|^2 = 1, |q| \leqq 1\}$$

modulo the equivalence relation $(a, b, c, q) \sim (a', b', c', q')$ if and only if for some $s \in Q$, $a = s \cdot a'$, $b = s \cdot b'$, $c = s \cdot c'$, and $q = q' \cdot s^{-1}$. The trivial 4-disc bundle over D^8 can be descirbed as $\{(a, b, c, q) \mid a, b, q \in Q, |a|^2 + |b|^2 = 1, |q| \leqq 1; c \in R, 0 \leqq c \leqq 1\}$ modulo the relation $(a, b, 0, q) \sim (c, d, 0, q)$ for any $a, b, c, d \in Q$. We may now describe a bundle map from this bundle to $\mathscr{S}^4$ over QP_2 by $p'([(a, b, c, q)]) = [(c \cdot a, c \cdot b, 1-c^2, q)]$, which is easily seen to be well defined on equivalence classes. This bundle map lies over the base space map $p: D^8 \to QP_2$, as may be seen by comparing the two descriptions of QP_2 which are used: the Thom space of the bundle $\mathscr{S}^4$ over QP_1 and the zero section of $\mathscr{S}^4$ over QP_2.

A second map from the trivial 4-disc bundle over D^8 to $\mathscr{S}^4$ over QP_2 is given by

$$\begin{aligned}(t \circ p)'([(a, b, c, q)]) &= [(c \cdot a, c \cdot b, \eta(a^{-1}b)(1-c^2), q \cdot \eta(a^{-1}b)^{-1})] \\ &= [(\eta(a^{-1}b)^{-1} \cdot c \cdot a, \eta(a^{-1}b)^{-1} \cdot c \cdot b, (1-c^2), q)].\end{aligned}$$

Note that this map can be thought of as the composition of p' and the bundle map over $t: QP_2 \to QP_2$ which takes "q" over $[(a, b, c)]$ to "q" over $t([(a, b, c)])$, and is thus a continuous bundle map over $t \circ p$.

Looking at our representation of $\mathscr{S}^4$ over QP_2 restricted to $S^4 \subset QP_2$, we see that the inverse image of the point $[(a, b, 0, q)]$ under p' is the set

$$\{[(q'a, q'b, 1, qq'^{-1})] \mid q' \in Q, |q'| = 1\},$$

and the inverse image of this same point under the map $(t \circ p)'$ is the set

$$\{[(q'a, q'b, 1, qq'^{-1}\eta(a^{-1}b))] \mid q' \in Q, |q'| = 1\}.$$

Thus, if we take two copies of the trivial 4-disc bundle over D^8 and identify the point represented by $[(a, b, 1, q)]$ in one with the point represented by

$$[(a, b, 1, q \cdot \eta(a^{-1}b))]$$

in the other, we get a bundle ρ over $S^8 = D^8 \cup D^8$, and the maps p' and $(t \circ p)'$ give a bundle map from this bundle to $\mathscr{S}^4$ over QP_2 which lies over p on one hemisphere, and over $t \circ p$ on the other. That is, there is a bundle map from ρ to $\mathscr{S}^4$ which lies over the element of $\pi_8(QP_2)$ we wish to compute.

The bundle ρ has coordinate transition map $S^7 \to SO(4)$ given by a composition $S^7 \to S^4 \to S^3 \to SO(4)$ which takes (a, b) to $f_{(a,b)}$ where $f_{(a,b)}(q) = q \cdot \eta(a^{-1}b)$ with quaternion multiplication indicated on the right. This map in $\pi_7(SO(4))$ is nontrivial (Steenrod [14, §22]), and thus ρ is a nontrivial bundle, and the map $S^8 \to QP_2$ is not nullhomotopic. Thus we have proven that the obstruction to finding a homotopy of t: $QP_2 \to QP_2$ to identity which is stationary on $S^4 \subset QP_2$ is nonzero.

The proof of Theorem 4 will now be completed by showing that t: $QP_2 \to QP_2$ is in fact homotopic to identity, and thus that the class of identity is the only homotopy class of maps $QP_2 \to QP_2$ which contains a tangential homotopy equivalence. Recall that t comes from a bundle map of $\mathscr{S}^4$ over $S^4 = QP_1$ to itself which lies over identity: $S^4 \to S^4$. The principal bundle associated with $\mathscr{S}^4$ is $\mathscr{S}^3_0$, with fiber and group S^3 = the group of unit length quaternions. Since $\mathscr{S}^3_0$ over S^4 has total space S^7, $\mathscr{S}^4$ is a universal bundle for disc bundles with group S^3 over complexes of dimension less than or equal to six [14]. Thus, there is a homotopy of t to the identity, T: $\mathscr{S}^4 \times I \to \mathscr{S}^4$, a bundle map which is t on $\mathscr{S}^4 \times (0)$ and identity on $\mathscr{S}^4 \times (1)$. Since T maps $\mathscr{S}^3_0 \times I$ to $\mathscr{S}^3_0$, this induces a map $QP_2 \times I \to QP_2$ which is the required homotopy of t to identity.

4. **Combinatorial equivalence.** In this section the results of the preceding two sections are applied to investigate $\theta(QP_n)$ for all $n \geqq 2$.

LEMMA 4.1. *Suppose (M, f) and (N, g) represent the same element in $\theta(QP_n)$ for some $n \geqq 2$, and that g: $N \to QP_n$ is homotopic to a map h: $N \to QP_n$ which is induced by a combinatorial equivalence. Then f: $M \to QP_n$ is also homotopic to a map induced by a combinatorial equivalence.*

Proof. Suppose τ_1: $K_1 \to N$, τ_2: $K_2 \to QP_n$, and c: $K_1 \to K_2$ give a combinatorial equivalence which induces h. Recall from Lemma 2.1 that there is a diffeomorphism

$d: N \to M$ such that g is homotopic to $f \circ d$. Then $d \circ \tau_1: K_1 \to M$ is a C^∞ triangulation of M. Let $h' = h \circ d^{-1}: M \to QP_n$. Then h' is induced by the combinatorial equivalence given by $d \circ \tau_1$, τ_2, and c. Further, $f \circ d$ homotopic to g implies f homotopic to h'.

LEMMA 4.2. *Let $\tau_1: K_1 \to M$ and $\tau_2: K_2 \to M$ be C^∞ triangulations of the manifold M. Then for any Riemannian metric on M and any $\varepsilon > 0$, there is a combinatorial equivalence $c: K_1 \to K_2$ such that the induced map $h = \tau_2 \circ c \circ \tau_1^{-1}$ is an ε-approximation to identity.*

Proof. Given $\varepsilon > 0$, choose $\varepsilon/2$ approximations $\tau_1': K_1' \to M$ and $\tau_2': K_2' \to M$ to τ_1 and τ_2 respectively where K_i' is a subdivision of K_i for $i = 1, 2$ as in [10, p. 101] such that for some combinatorial equivalence $c': K_1' \to K_2'$, $\tau_2' \circ c' \circ \tau_1'^{-1}: M \to M$ is identity. Since K_1' and K_2' are subdivisions of K_1 and K_2, c' induces a combinatorial equivalence $c: K_1 \to K_2$. Then $h = \tau_2 \circ c \circ \tau_1^{-1}$ is an ε-approximation to identity, and by definition is induced by c.

PROPOSITION 4.1. *If (M, f) is any representative of an element of $\theta(QP_2)$, then $f: M \to QP_2$ is homotopic to a map induced by a combinatorial equivalence.*

Proof. By Lemma 4.1, we need only prove the proposition for one representative in each equivalence class in $\theta(QP_2)$. The proposition is obvious for (QP_2, identity), so, by Theorem 4, we need only work with ($QP_2 \# \Sigma^8$, id$'$) where id$'$ is defined as in Lemma 2.2 by a map $\phi: (QP_2 - i(D^{\circ 8})) \cup_d D^8 \to (QP_2 - i(D^{\circ 8})) \cup_i D^8$ such that $\phi(x) = x$ for $x \in QP_2 - i(D^{\circ 8})$, and $\phi(s, \tau) = (d^{-1}(s), \tau)$ for $(s, \tau) \in D^8$, for $d: S^7 \to S^7$ a representative of the nonzero element of Γ_8.

We now refer to [11, p. 546] where for each n a C^∞-triangulation $g_n: K^n \to D^n$ is constructed having the property that if $h: S^{n-1} \to S^{n-1}$ is any map which induces a combinatorial equivalence of $\partial|K^n|$ with itself then the radial extension of h to a map $h_r: D^n \to D^n$ induces a combinatorial equivalence of K^n with itself.

Proceed as in [11]. Consider the triangulations of $\partial(QP_2 - i(D^{\circ 8}))$ given by restriction to the boundary of $i \circ g_8: K^8 \to QP_2 - i(D^{\circ 8})$, and of $d \circ i \circ g_8: K^8 \to QP_2 - i(D^{\circ 8})$. Each of these C^∞-triangulations has an extension over all of $QP_2 - i(D^{\circ 8})$ as in [10, p. 101], and, by Lemma 4.2, we may choose a combinatorial equivalence c of these triangulations such that the map h induced by this equivalence is homotopic to identity via a homotopy which maps $\partial(QP_2 - i(D^{\circ 8})) \times I$ to $\partial(QP_2 - i(D^{\circ 8}))$. Restricted to $\partial(QP_2 - i(D^{\circ 8}))$, h gives a combinatorial equivalence $c|\partial|K_1|$. Thus the radial extension of h to a map $h': (QP_2 - i(D^{\circ 8})) \cup_d D^8 \to (QP_2 - i(D^{\circ 8})) \cup_i D^8$ gives a combinatorial equivalence, and extending the homotopy of h to identity radially gives a homotopy of h' to id$'$ as needed.

PROPOSITION 4.2. *Suppose $f: M \to QP_n$ for $n \geqq 2$ is differentiable so that there is a natural differentiable structure on the pullback $f^*(\mathcal{S}^4)$, and that $p: f^*(\mathcal{S}^4) \to \mathcal{S}^4$ is a map induced by a combinatorial equivalence. Let $i: D^{4n+4} \to QP_{n+1}$ be an imbedding such that $i(D^{4n+4}) \cap (QP_n \subset QP_{n+1}) = \varnothing$, and identify QP_{n+1} with*

$\mathscr{S}^4 \cup_i D^{4n+4}$. If d: $S^{4n+3} \to \partial f^*(\mathscr{S}^4)$ is a diffeomorphism, and we form the manifold $f^*(\mathscr{S}^4) \cup_d D^{4n+4}$ and extend p to p_d: $f^*(\mathscr{S}^4) \cup_d D^{4n+4} \to \mathscr{S}^4 \cup_i D^{4n+4}$ by $p_d(x) = p(x)$ for $x \in f^*(\mathscr{S}^4)$, $p_d(s, \tau) = (i^{-1}(p(d(s))), \tau)$ for $(s, \tau) \in D^{4n+4}$, then p_d is homotopic to a map p_d': $f^*(\mathscr{S}^4) \cup_d D^{4n+4} \to \mathscr{S}^4 \cup_i D^{4n+4}$ which is induced by a combinatorial equivalence.

Proof. Again we will use Munkres' triangulation g_{4n+4}: $K^{4n+4} \to D^{4n+4}$ and proceed as in [11]. $d \circ g_{4n+4}$ and $p \circ d \circ g_{4n+4}$ are respectively C^∞-triangulations of $\partial f^*(\mathscr{S}^4)$ and $\partial \mathscr{S}^4$. These extend to triangulations τ_3: $K_3 \to f^*(\mathscr{S}^4)$ and τ_4: $K_4 \to \mathscr{S}^4$ respectively, as in [10, p. 101]. By Lemma 4.2, there are combinatorial equivalences c_1 and c_2 such that in the following diagram i_1 and i_2 are approximations to the identity maps close enough so that i_1 and i_2 are each homotopic to identity via homotopies which always carry $\partial f^*(\mathscr{S}^4)$ to $\partial f^*(\mathscr{S}^4)$ and $\partial \mathscr{S}^4$ to $\partial \mathscr{S}^4$. In the diagram, c is a combinatorial equivalence, and $p = \tau_2 \circ c \circ \tau_1^{-1}$, where τ_1 and τ_2 are C^∞-triangulations.

$$\begin{array}{ccccccc}
f^*(\mathscr{S}^4) & \xrightarrow{i_1} & f^*(\mathscr{S}^4) & \xrightarrow{p} & \mathscr{S}^4 & \xrightarrow{i_2} & \mathscr{S}^4 \\
\downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_4} \\
K_3 & \xrightarrow{c_1} & K_1 & \xrightarrow{c} & K_2 & \xrightarrow{c_2} & K_4
\end{array}$$

Restricted to $\partial f^*(\mathscr{S}^4)$, the map $i_2 \circ p \circ i_1$ induces the equivalence $c_2 \circ c \circ c_1$ on ∂K^{4n+4}, and thus the radial extension of $i_2 \circ p \circ i_1$ induces a combinatorial equivalence $K^{4n+4} \to K^{4n+4}$. So, triangulating D^{4n+4} by g_{4n+4}, we get

$$\begin{array}{ccc}
f^*(\mathscr{S}^4) \cup_d D^{4n+4} & \xrightarrow{p_d'} & \mathscr{S}^4 \cup_i D^{4n+4} \equiv QP_{n+1} \\
\downarrow{\scriptstyle \tau_3'} & & \downarrow{\scriptstyle \tau_4'} \\
K_3' & \xrightarrow{c'} & K_4'
\end{array}$$

where c' is a combinatorial equivalence and τ_3', τ_4' are C^∞-triangulations and p_d' is the radial extension of $i_2 \circ p \circ i_1$. But p_d' is homotopic to the radial extension of p via radial extensions of the homotopies of i_1 and i_2 to identity maps, and the proposition is proven.

LEMMA 4.3. *Suppose f: $\sigma^p \to R^n$ is a map of a p-simplex into R^n such that for some subdivision of $\partial\sigma$, $f|\partial\sigma$ is differentiable on each simplex of $\partial\sigma$. Then f is homotopic to a map f': $\sigma^p \to R^n$ such that $f'|\partial\sigma = f|\partial\sigma$, and for some subdivision of σ, f' is differentiable on each simplex in σ.*

Proof. Assume $\sigma^p \subset R^p$ such that b_σ = barycenter of σ is at the origin of R^p. Let l: $R^p \to R^p$ be defined by $l(x) = x/2$, and let $N = \sigma - l(\sigma)$. Subdivide σ and its faces so that $f|\partial\sigma$ is differentiable on each simplex of $\partial\sigma$ and triangulate $l(\partial\sigma)$ by the composition of this triangulation of $\partial\sigma$ and l. Sets of the form $\{x \in N | \tau \cdot x \in \gamma$

for some τ, $1 \leqq \tau \leqq 2\}$ as γ ranges over the simplices of $\partial\sigma$, the simplices of $\partial\sigma$, and the images of the simplices of $\partial\sigma$ under l give a cellular decomposition of N [10, p. 70].

As in [10], this cell complex may be divided into a simplicial complex without subdividing any of the cells of $\partial N = \partial\sigma \cup l(\partial\sigma)$, since these cells are already simplices. Define $f'\colon \partial N \to R^n$ by $f'|\partial\sigma = f|\partial\sigma$, and $f'|l(\partial\sigma)$ is that map which is linear on each simplex of $l(\partial\sigma)$, and such that f and f' agree on the vertices of $l(\partial\sigma)$. Given $x \in N$, let $\tau(x)$ be the real number between 1 and 2 such that $\tau(x)\cdot x \in \partial\sigma$. Then define f' on the interior of N by

$$f'(x) = (2-\tau(x))\cdot f(\tau(x)\cdot x) + (\tau(x)-1)\cdot f'(\tau(x)\cdot x/2).$$

Then f' is differentiable on each simplex of the triangulation of N, and f' is linear on each simplex in $\partial(l(\sigma))$. We triangulate $l(\sigma)$ by taking the join of b_σ with the simplices in $\partial(l(\sigma))$, and extend f' linearly over the simplices in $l(\sigma)$, letting $f'(b_\sigma) = f(b_\sigma)$, to get the required map $f'\colon \sigma \to R^n$.

LEMMA 4.4. *Let σ^p be a simplex in Euclidean space. Then any C^∞-triangulation of $\partial(\sigma) \times D^k$ has a subdivision which can be extended to a C^∞-triangulation of $\sigma \times D^k$.*

Proof. The proof is in two parts; first extend a subdivision to a triangulation of $\partial(\sigma \times D^k)$, and then extend a subdivision of this to the needed triangulation of $\sigma \times D^k$.

Suppose $\tau\colon K \to \partial(\sigma) \times D^k$ is a C^∞-triangulation. $\partial(\sigma \times D^k) = \partial(\sigma) \times D^k \cup \sigma \times \partial D^k$, where the intersection $\partial\sigma \times \partial D^k$ already has a C^∞-triangulation then. Subdivide K so that in this triangulation of $\partial\sigma \times \partial D^k$ every simplex γ of the subdivided complex which is mapped by τ into $\partial\sigma \times \partial D^k$ is mapped into a face of σ by (projection $\circ\, \tau$): $K \to \partial\sigma$.

Let $r\colon \sigma^p \to D^p$ be a map as in [11, p. 546] which is a C^∞-triangulation of D^p if we take the first barycentric subdivision of σ, and define a map $(\partial\sigma) \times (\partial D^k) \to S^{p-1} \times S^{k-1}$ by $(x, v) \to (r(x), v)$. This induces a C^∞-triangulation of $S^{p-1} \times S^{k-1}$ which can be extended [10] to a C^∞-triangulation of $D^p \times S^{k-1}$. We further subdivide this extension so that it pulls back to a C^∞-triangulation of $\partial(\sigma \times D^k)$ which extends a subdivision of K. Denote this triangulation of $\partial(\sigma \times D^k)$ by

$$\tau'\colon K' \to \partial(\sigma \times D^k).$$

Let b_σ denote the barycenter of σ, and 0 the center of D^k. For some integer $s \geqq 2$, the distance from b_σ to $\partial\sigma$ is greater than $2/s$. For $(x, y) \in \sigma \times D^k - (b_\sigma, 0)$, let $p(x, y)$ denote the unique point of $\partial(\sigma \times D^k)$ which lies on the ray from $(b_\sigma, 0)$ through (x, y), and $d(x, y)$ the distance from (x, y) to $p(x, y)$. Let

$$N = \{(x, y) \in \sigma \times D^k \mid d(x, y) \leqq 1/s\},$$

and define $p_1\colon N \to \partial(\sigma \times D^k) \times [0, 1/s]$ by $p_1(x, y) = (p(x, y), d(x, y))$. Then p_1 is a homeomorphism, and

(1) for γ any face of σ, $p_1|p_1^{-1}(\gamma \times D^k)$ is a C^∞-map of rank equal to $\dim \gamma + k$,
(2) $p_1|p_1^{-1}(\sigma \times \partial D^k)$ is a C^∞-map of rank $p+k-1$.

Proceed as in [10, p. 102]. Subdivide K' to a complex K^2 such that the image of each simplex of K^2 lies in a subset of $\partial(\sigma \times D^k)$ of the form $\gamma \times D^k$ for γ a face of σ, or in $\sigma \times \partial D^k$. Then triangulate $\partial(\sigma \times D^k) \times [0, 1/s]$ by first triangulating $\partial(\sigma \times D^k) \times I$ as in [10, p. 102] and then mapping $\partial(\sigma \times D^k) \times I \to \partial(\sigma \times D^k) \times [0, 1/s]$ by $(x, \tau) \to (x, \tau/s)$. Then p_1^{-1} composed with this triangulation of $\partial(\sigma \times D^k) \times [0, 1/s]$ gives a C^∞-triangulation of N which extends $\tau' \colon K^2 \to \partial(\sigma \times D^k)$. As in [10], we fit this triangulation of N together with a C^∞-triangulation of interior $(\sigma \times D^k)$ to get the needed triangulation of $\sigma \times D^k$.

PROPOSITION 4.3. *Assume $f \colon M \to QP_n$ is homotopic to a map h induced by a combinatorial equivalence. Then f is homotopic to a differentiable map $f_1 \colon M \to QP_n$ such that the natural map $\bar{f}_1 \colon f_1^*(\mathcal{S}^4) \to \mathcal{S}^4$ is homotopic to a map induced by a combinatorial equivalence, and the homotopy may be taken so that it always maps $\partial(f_1^*(\mathcal{S}^4))$ to $\partial(\mathcal{S}^4)$.*

Proof. We may choose trivialization of $\mathcal{S}^4$ over a finite number, say m, of coordinate neighborhoods in QP_n such that

(1) the trivializations are given by maps $\tau_i \colon D^{\circ 4n} \times D^4 \to \mathcal{S}^4$ for $1 \leqq i \leqq m$, where $D^{\circ 4n} = \{x \in D^{4n} \mid |x| < 1\}$, and the induced base space maps $\tau_i | D^{\circ 4n} \times (0)$ are diffeomorphisms onto their images which we denote by $0_i \subset QP_n$.

(2) $QP_n \subset \bigcup_{i=1}^m \tau_i(D_{1-\varepsilon}^{4n} \times (0))$ for some $\varepsilon > 0$ where $D_{1-\varepsilon}^{4n} = \{x \in D^{4n} \mid |x| \leqq 1 - \varepsilon\}$.

(3) the coordinate bundle transition maps in $\mathcal{S}^4$ induced by the τ_i, $g_{ij} \colon 0_i \cap 0_j \to SO(4)$ are C^∞-maps for all i, j, $1 \leqq i, j \leqq m$.

Assume $\tau_1 \colon K_1 \to M$ and $\tau_2 \colon K_2 \to QP_n$ are C^∞-triangulations, and $c \colon K_1 \to K_2$ is a combinatorial equivalence such that $h = \tau_2 \circ c \circ \tau_1^{-1}$. Since f is homotopic to h, f is homotopic to a differentiable map f_1 which is an $\varepsilon/5$ approximation to h in some Riemannian metric on QP_n for which the trivialization maps τ_i all induce distance increasing maps on base spaces. Assume K_1 and K_2 are subdivided so that $c \colon K_1 \to K_2$ is a simplicial isomorphism, and thus $h \circ \tau_1 \colon K_1 \to QP_n$ is a C^∞-triangulation.

For any i, $1 \leqq i \leqq m$, $h^{-1}(0_i) \cap f_1^{-1}(0_i)$ contains $h^{-1}(\tau_i(D_{1-\varepsilon/2}^{4n} \times (0))$, and hence sets of the form $h^{-1}(0_i) \cap f_1^{-1}(0_i)$ form an open cover of M. We assume K_1 has been subdivided so that for each simplex σ of K_1 there is an i such that

$$\tau_1(\sigma) \subset h^{-1}(0_i) \cap f_1^{-1}(0_i).$$

We pull the trivialization we have selected of $\mathcal{S}^4$ over 0_i back to a trivialization of $h^*(\mathcal{S}^4)$ over $h^{-1}(0_i)$ and of $f_1^*(\mathcal{S}^4)$ over $f_1^{-1}(0_i)$ for each i. Given any $4n$-simplex σ of K_1, trivialize $h^*(\mathcal{S}^4) | \tau_1(\sigma)$ and $f_1^*(\mathcal{S}^4) | \tau_1(\sigma)$ by fixing some choice of i such that $\tau_1(\sigma) \subset f_1^{-1}(0_i) \cap h^{-1}(0_i)$ and using the corresponding trivialization over $\tau_1(\sigma)$. This induces a trivialization of $h^*(\mathcal{S}^4) | \tau_1(\gamma)$ and of $f_1^*(\mathcal{S}^4) | \tau_1(\gamma)$ for any face γ of σ, and in this fashion fixes trivializations of $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ over each of the simplices of the triangulation of M given by $\tau_1 \colon K_1 \to M$.

Let $H \colon M \times I \to QP_n$ be a homotopy with $H | M \times (0) = h$, $H | M \times (1) = f_1$, and identify $H^*(\mathcal{S}^4)|M\times(0)$ with $h^*(\mathcal{S}^4)$ and $H^*(\mathcal{S}^4)|M\times(1)$ with $f_1^*(\mathcal{S}^4)$. Then a bundle isomorphism $H^*(\mathcal{S}^4)\approx h^*(\mathcal{S}^4)\times I$ gives us a bundle isomorphism

$$b_1: h^*(\mathcal{S}^4)\to f_1^*(\mathcal{S}^4)$$

over identity: $M\to M$ such that if $\bar{h}: h^*(\mathcal{S}^4)\to\mathcal{S}^4$ and $\bar{f}_1: f_1^*(\mathcal{S}^4)\to\mathcal{S}^4$ are the natural maps then $\bar{h}b_1^{-1}$ is homotopic as a bundle map to $\bar{f}_1$.

Given an arbitrary simplex σ of K_1, it is possible that σ is a face of several $4n$-simplices, and thus that $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ are each trivialized above in several ways over $\tau_1(\sigma)$. If specific trivializations of $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ are fixed over $\tau_1(\sigma)$, b_1 induces a map $\sigma\times D^4\to\sigma\times D^4$ which is represented by a map $\sigma\to SO(4)$. If trivializations pulled back from the trivialization τ_i of $\mathcal{S}^4$ over 0_i are used, we will denote this induced map by $b_\sigma^i:\sigma\to SO(4)$. If trivializations pulled back from τ_j over 0_j are used, we get b_σ^j where $b_\sigma^j=(g_{ij}\circ f_1\circ\tau_1)\cdot b_\sigma^i\cdot(g_{ji}\circ h\circ\tau_1)$ where the dots indicate composition of elements in $SO(4)$. Thus if b_σ^i is a C^∞-map, then b_σ^j is also C^∞.

Subdivide K_1 so that for all simplices $\gamma\in K_1$ and all i, $b_\gamma^i(\gamma)$ is contained in a coordinate neighborhood U_γ^i in the differentiable structure of $SO(4)$. For each $\gamma\in K_1$, and all i, choose a diffeomorphism $d_\gamma^i: U_\gamma^i\to R^6$. We smooth the map b_1 as follows:

Suppose $p\geqq 1$ and b_1 is such that all of the induced maps b_σ^i are C^∞ for all simplices σ of K_1 of dimension less than or equal to $p-1$. Let γ be a p-simplex of K_1. Choose some i, $1\leqq i\leqq m$ such that γ is a p-face of a $4n$-simplex σ where $\tau_1(\sigma)\subset h^{-1}(0_i)\cap f_1^{-1}(0_i)$, and thus fix a choice of one of our trivializations of $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ over $\tau_1(\gamma)$. For any face α of γ, $b_\gamma^i|\alpha:\alpha\to SO(4)$ is C^∞ by assumption, and so $d_\gamma^i\circ b_\gamma^i:\alpha\to R^6$ is C^∞ for all faces α of γ. By Lemma 4.3, we may choose a map $c_\gamma^i:\gamma\to R^6$ homotopic to $d_\gamma^i\circ b_\gamma^i$ such that $c_\gamma^i|\partial\gamma=d_\gamma^i\circ b_\gamma^i|\partial\gamma$ and c_γ^i is a C^∞-map on each simplex of some subdivision of γ. Then $d_\gamma^{i-1}\circ c_\gamma^1$: $\gamma\to SO(4)$ is equal to b_γ^i on $\partial\gamma$ and differentiable on each simplex of some subdivision of γ. Thus b_1 can be deformed by a bundle homotopy which is stationary over the image under τ_1 of the p-skeleton of K_1 minus γ so that the induced map of the resulting bundle map over $\tau_1(\gamma)$ in the trivializations of $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ over $\tau_1(\gamma)$ corresponding to i is $d_\gamma^{i-1}\circ c_\gamma^i$. Do this for all p-simplices γ of K_1 to get a bundle map which we again denote by b_1 such that the induced maps b_γ^i are all C^∞-maps on some subdivision of γ for all γ of dimension less than or equal to p. Since all maps b_σ^i are in fact C^∞ for σ of dimension 0, we see by induction that we may subdivide K_1 to a complex K and deform b_1 by a series of bundle homotopies to a map b such that all induced maps $b_\sigma^i:\sigma\to SO(4)$ are C^∞ for all $\sigma\in K$, $1\leqq i\leqq m$.

Since K is a subdivision of K_1, each $4n$-simplex of K is contained in a unique $4n$-simplex of K_1, and so our choices of trivializations of $h^*(\mathcal{S}^4)$ and $f_1^*(\mathcal{S}^4)$ over the simplices $\tau_1(\sigma)$ for $\sigma\in K$, induce choices of trivializations over the simplices $\tau_1(\gamma)$ for $\gamma\in K$. Let γ be a simplex of K. We will need the following facts:

(1) If γ is a face of two different $4n$-simplices σ_1 and σ_2 of K, the bundle

$h^*(\mathscr{S}^4)|\tau_1(\gamma)$ is trivialized above in two possibly distinct ways. The map induced by this $\gamma \times D^4 \to \gamma \times D^4$ is a diffeomorphism.

Proof. This map is a diffeomorphism if and only if the map induced $\gamma \to SO(4)$ is a C^∞-map. If coordinates over $\tau_1(\sigma_1)$ come from the trivializing map τ_i and coordinates over $\tau_1(\sigma_2)$ come from τ_j, the map $\gamma \to SO(4)$ is $g_{ij} \circ h \circ \tau_1 : \gamma \to SO(4)$. Since $h \circ \tau_1 : K \to QP_n$ is a C^∞-triangulation, we may extend $g_{ij} \circ h \circ \tau_1$ to a C^∞-map on a neighborhood $N(\gamma)$ of γ in Euclidean space, and the map $\gamma \times D^4 \to \gamma \times D^4$ is a diffeomorphism.

(2) Using one of our choices of trivialization of $h^*(\mathscr{S}^4)|\tau_1(\gamma)$ we map $\gamma \times D^4$ to $h^*(\mathscr{S}^4)|\tau_1(\gamma)$. Then the composition

$$\gamma \times D^4 \longrightarrow h^*(\mathscr{S}^4)|\tau_1(\gamma) \xrightarrow{\bar{h}} \mathscr{S}^4$$

is a diffeomorphism onto its image where $\bar{h}$ is the natural map.

Proof. Since $h \circ \tau_1|\gamma : \gamma \to QP_n$ is a C^∞-map of rank equal to the dimension of γ, and we have chosen τ_i so that the map $0_i \times D^4 \to \mathscr{S}^4$ is a diffeomorphism onto its image, this is clear.

(3) Suppose for some simplex σ and one of our choices of trivialization of $h^*(\mathscr{S}^4)|\tau_1(\gamma)$, we have a map $\sigma \to h^*(\mathscr{S}^4)|\tau_1(\gamma)$ such that the composition $\sigma \to \gamma \times D^4$ is a C^∞-map of rank equal to the dimension of σ. Then, by (1) above, for any of the other choices of trivialization of $h^*(\mathscr{S}^4)|\tau_1(\gamma)$ fixed above, the composition $\sigma \to \gamma \times D^4$ is a C^∞-map of rank equal to dim σ.

We will call a triangulation of $h^*(\mathscr{S}^4)$ differentiable if the image of each simplex σ of the triangulation is contained in $h^*(\mathscr{S}^4)|\tau_1(\gamma)$ for some $\gamma \in K$, and the triangulation is such that the induced map $\sigma \to \gamma \times D^4$ for at least one, and thus by (3) all, of our choices of trivialization of $h^*(\mathscr{S}^4)|\tau_1(\gamma)$ is a C^∞-map of rank equal to dim σ. We now construct such a triangulation of $h^*(\mathscr{S}^4)$ inductively.

Assume $h^*(\mathscr{S}^4)|\tau_1(K^p)$, where $K^p =$ the p-skeleton of K, is differentiably triangulated in this sense by $\tau_p' : K'^p \to h^*(\mathscr{S}^4)|\tau_1(K^p)$. (Note then dim $K'^p = p+4$.) Let σ^{p+1} be any $(p+1)$-simplex of K, and choose one of the trivializations we have fixed for $h^*(\mathscr{S}^4)|\tau_1(\sigma)$. Then $h^*(\mathscr{S}^4)|\tau_1(\partial\sigma)$ is already triangulated by the restriction of τ_p' to a subcomplex $K_\sigma'^p$ of K'^p. Using the trivialization map $h^*(\mathscr{S}^4)|\tau_1(\sigma) \approx \sigma \times D^4$, we get a C^∞-triangulation $K_\sigma'^p \to (\partial\sigma) \times D^4$. By Lemma 4.4 for $k=4$, there is a differentiable triangulation of $\sigma \times D^4$ which extends a subdivision of $\tau_p' : K_\sigma'^p \to (\partial\sigma) \times D^4$. Thus there is a complex $\bar{K}_\sigma^p$ which contains a subdivision of $K_\sigma'^p$ and a map $\bar{\tau}_p : \bar{K}_\sigma^p \to h^*(\mathscr{S}^4)|\tau_1(\sigma)$ which extends τ_p' and which is a differentiable triangulation of $h^*(\mathscr{S}^4)|\tau_1(\sigma)$ in the above defined sense. We adjoint $\bar{K}_\sigma^p$ to a subdivision of K'^p by identifying subdivisions of $K_\sigma'^p$ so that the resulting complex gives a differentiable triangulation of $h^*(\mathscr{S}^4)|\tau_1(K^p \cup \sigma)$.

Proceeding in this fashion for each $(p+1)$-simplex σ of K, we obtain a complex K'^{p+1} which contains a subdivision of K'^p, and a map

$$\tau_{p+1}' : K'^{p+1} \to h^*(\mathscr{S}^4)|\tau_1(K^{p+1})$$

which extends τ'_p and which is a differentiable triangulation of $h^*(\mathscr{S}^4)|\tau_1(K^{p+1})$. Since $h^*(\mathscr{S}^4)|\tau_1(K^0)$ may easily be triangulated differentiably, we see by induction that there is a differentiable triangulation $\tau' \colon K' \to h^*(\mathscr{S}^4)$.

For any simplex σ' of K', then, there is some $\sigma \in K$ such that $\tau'(\sigma') \subset h^*(\mathscr{S}^4)|\tau_1(\sigma)$, and for any of the trivializations of $h^*(\mathscr{S}^4)|\tau_1(\sigma)$, we have fixed, the induced map $\sigma' \to \sigma \times D^4$ is a C^∞-map of rank equal to dim σ'. By property (2) above of the trivialization maps, $\bar{h} \circ \tau'|\sigma'$ is a C^∞-map of rank equal to dim σ' also, and hence $\bar{h} \circ \tau' \colon K' \to \mathscr{S}^4$ is a C^∞-triangulation of $\mathscr{S}^4$.

Again, for σ' a simplex of K', we choose $\sigma \in K$ such that $\tau'(\sigma') \subset h^*(\mathscr{S}^4)|\tau_1(\sigma)$. If trivializations in $h^*(\mathscr{S}^4)$ and $f_1^*(\mathscr{S}^4)$ induced from τ_i in $\mathscr{S}^4$ over 0_i are used, then the induced maps $\sigma' \to h^*(\mathscr{S}^4) \to \sigma \times D^4$ and thus

$$\sigma' \to \sigma \times D^4 \to \sigma \times D^4 \to f_1^*(\mathscr{S}^4)$$

are C^∞ and of rank equal to dim σ', where the map $\sigma \times D^4 \to \sigma \times D^4$ is given by $(x, v) \to (x, b_\sigma^i(x) \cdot v)$ and $\sigma \times D^4 \to f_1^*(\mathscr{S}^4)$ is the trivialization map. But this is just $b \circ \tau'|\sigma'$, so $b \circ \tau' \colon K' \to f_1^*(\mathscr{S}^4)$ is a C^∞-triangulation.

Thus $\bar{h} \circ \tau' \colon K' \to \mathscr{S}^4$ and $b \circ \tau' \colon K' \to f_1^*(\mathscr{S}^4)$ are C^∞-triangulations, and $\bar{h}b^{-1} \colon f_1^*(\mathscr{S}^4) \to \mathscr{S}^4$ is a map induced by a combinatorial equivalence, identity: $K' \to K'$. $\bar{h}b^{-1}$ is homotopic to $\bar{h}b_1^{-1}$ which is homotopic to $\bar{f}_1$, so the proposition is proven.

The proof of Theorem 5 can now be given: Proposition 4.1 is just Theorem 5 for the case $n=2$. We proceed by induction on n.

Assume Theorem 5 is known true for all m satisfying $2 \leqq m \leqq r-1$, and that (M, f) represents an element of $\theta(QP_r)$. By the results of §2, $(M, f) \sim (N, g)$ in $\theta(QP_r)$ where $g \colon N \to QP_r$ is differentiable and transverse regular over $QP_{r-1} \subset QP_r$ and $[(g^{-1}(QP_{r-1}), g \mid g^{-1}(QP_{r-1}))] = r([(M, f)])$. Then

$$(S, \tau) \equiv (g^{-1}(QP_{r-1}), g \mid g^{-1}(QP_{r-1}))$$

satisfies the theorem, and by Proposition 4.3, τ is homotopic to a differentiable map $\tau_1 \colon S \to QP_{r-1}$ such that $\bar{\tau}_1 \colon \tau_1^*(\mathscr{S}^4) \to \mathscr{S}^4$ is homotopic as a bundle map to a map induced by a combinatorial equivalence. $[(S, \tau)] = [(S, \tau_1)] \in$ image (r) implies $\partial(\tau_1^*(\mathscr{S}^4)) = \tau_1^*(\mathscr{S}_0^3)$ is diffeomorphic to S^{4r-1}, and if we choose a diffeomorphism d and attach D^{4r} to $\tau_1^*(\mathscr{S}^4)$ using d then $\bar{\tau}_1$ extends radially to a tangential homotopy equivalence $\bar{\tau} \colon \tau_1^*(\mathscr{S}^4) \cup_d D^{4r} \to QP_r$. By Proposition 4.2, $\bar{\tau}$ is homotopic to a map induced by a combinatorial equivalence.

Since $r([(\tau_1^*(\mathscr{S}^4) \cup_d D^{4r}, \bar{\tau})]) = r([(N, g)])$, there is some choice of attaching diffeomorphism d such that $[(N, g)] = [(\tau_1^*(\mathscr{S}^4) \cup_d D^{4r}, \bar{\tau})]$ in $\theta(QP_r)$. Since $(M, f) \sim (N, g)$, Lemma 4.1 shows $f \colon M \to QP_r$ is homotopic to a map induced by a combinatorial equivalence. Hence Theorem 5 is proven for all integers m satisfying $2 \leqq m \leqq r$, and by induction the theorem is proven.

Note. It seems likely that the more sophisticated techniques of Sullivan [15] could be applied to prove Theorem 5.

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